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Entropy theory for locally compact sofic groups

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Entropy theory for locally compact sofic groups

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Dedicated to the loving memory of my Beeji.

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Entropy theory for locally compact sofic groups

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In the past decade entropy theory for the actions of countable sofic groups has been developed starting with the work of Bowen[Bow10b][Bow10c] and its extension by Kerr-Li [KL11]. We extend their work by introducing locally compact sofic groups and developing entropy theory for actions of locally compact sofic groups – thereby producing measurable and topological dynamical invariants and establishing the variational principle. We compute the entropy for Poisson point processes on sofic groups and further establish the relationship between the entropies of an action of a group and its restriction to a lattice subgroup.

Contents

Chapter 1

Introduction

Kolmogorov [Kol58] [Kol59] and Sinai [Sin59] introduced entropy as an invariant for automorphisms of a probability measure space. This invariant was introduced as a quantification of unpredictability of a dynamical system with the belief that it would aid in distinguishing "deterministic" dynamical systems from "probabilistic" ones. Since then, entropy has become one of the most important tools in the classification theory of dynamical systems, as well as several other problems in ergodic theory.

Let $(X, \mu, \mathscr{B}, T)$ be a dynamical system where $(X, \mu, \mathscr{B})$ is a standard probability space, i.e. it is isomorphic – as a measure space – to the unit interval with Lebesgue measure, and $T : (X, \mu, \mathscr{B}) \to (X, \mu, \mathscr{B})$ is a measure space isomorphism. We say that the two systems $(X, \mu, \mathscr{B}, T)$ and $(Y, \nu, \mathscr{C}, S)$ are **isomorphic** if there is a measure space isomorphism $\Psi : (X, \mu, \mathscr{B}) \to (Y, \nu, \mathscr{C})$ such that $\Psi \circ T = S \circ \Psi$ almost everywhere.

If $\mathcal{P}$ is a countable partition of X comprised of pairwise disjoint pieces, then the **Shannon entropy** of $\mathcal{P}$ is defined as

$$H_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P).$$

The pullback of $\mathcal{P}$ is the partition

$$T^{-1}\mathcal{P} = \{T^{-1}P : P \in \mathcal{P}\ \}$$

and the refinement of the two partitions $\mathcal{P}$ and $\mathcal{Q}$ is defined as

$$\mathcal{P} \vee \mathcal{Q} = \{P \cap Q \mid P \in \mathcal{P}, Q \in \mathcal{Q}\}.$$

The entropy rate of T with respect to $\mathcal{P}$ is

$$h_\mu(T,\mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n} T^{-i}\mathcal{P}\right)$$

which can be viewed as a quantification of the unpredictability per unit time when observing a random point x at the scale of $\mathcal{P}$. The Kolmogorov-Sinai entropy, or **entropy rate** of the system $(X,\mu,\mathscr{B},T)$, is defined as

$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T),$$

where the supremum is over all partitions with finite Shannon entropy.

By virtue of its definition, this entropy is clearly an isomorphism invariant of the system $(X,\mu,\mathscr{B},T)$. But, this definition does not lend itself well to the calculations. Sinai [Sin59] provided the main tool for computing $h_\mu(T)$ by using generating partitions: a partition $\mathcal{P}$ is called **generating** for the system $(X,\mu,\mathscr{B},T)$ if the smallest σ- algebra (up to measure zero) containing $\{T^n P : P \in \mathcal{P} \text{ and } -\infty < n < \infty\}$ is $\mathcal{B}$. This allows one to calculate $h_\mu(T)$ using any generating partition:

Theorem 1.0.1 (Sinai 59)**.** *If $\mathcal{P}$ is a generating partition for the system $(X, \mu, \mathscr{B}, T)$ with finite Shannon entropy, then*

$$h_\mu(T) = h_\mu(T, \mathcal{P}).$$

The definitions above were largely motivated to understand Bernoulli shifts, which have since played a central role in the development of entropy theory. To define a Bernoulli shift, suppose (K, κ) is a finite set with the probability measure κ and let $(K^{\mathbb{Z}}, \kappa^{\mathbb{Z}})$ be the product measure space. The Bernoulli shift $\sigma : (K^{\mathbb{Z}}, \kappa^{\mathbb{Z}}) \to (K^{\mathbb{Z}}, \kappa^{\mathbb{Z}})$ is defined by $\sigma(x)_n := x_{n-1}$ whenever $x = \{x_n\}_{n\in\mathbb{Z}} \in K^{\mathbb{Z}}$. The partition $\mathcal{P} = \{P_k : k \in K\}$ where $P_k = \{x \in K^{\mathbb{Z}} : x_0 = k\}$ is generating for the system $(K^{\mathbb{Z}}, \kappa^{\mathbb{Z}}, \sigma)$ and, the Kolmogorov-Sinai entropy of $(K^{\mathbb{Z}}, \kappa^{\mathbb{Z}}, \sigma)$ is the same as the Shannon entropy of the base space (K, κ). That is

$$h_\mu(\sigma) = -\sum_{k\in K} \kappa(k) \log \kappa(k).$$

For example, entropy of the Bernoulli shift over the two point space $(\{0, 1\}, \kappa_2)$ with the uniform probability measure κ_2 is $\log 2$. However, the shift over the three point space $(\{0, 1, 2\}, \kappa_3)$ with the uniform probability measure κ_3, is $\log 3$; hence, the two cannot be isomorphic.

There is a natural generalization of Bernoulli shifts over the group of integers $\mathbb{Z}$ to Bernoulli shifts over any countable group Γ. As before, let (K, κ) be a finite probability space and $(K^\Gamma, \kappa^\Gamma)$ be the product space. The group Γ acts on the product space by automorphisms $\gamma \,.\, x(\gamma') = x(\gamma^{-1}\gamma')$ where $K^\Gamma \ni x : \Gamma \to K$ and $\gamma, \gamma' \in \Gamma$.

Thus, the instinctive question is: can the above framework of entropy be extended to the actions of groups beyond integers?

The first possible extension is to the actions of amenable groups, indeed, by almost exactly the same definition. Let us recall that a locally compact second countable group G is **amenable** if there exists a sequence of compact sets $\{F_i\}_{i=1}^{\infty} \subset G$, called Følner sets, such that for any compact set $K \subset G$,

$$\lim_{i\to\infty} \frac{\lambda(KF_i \Delta F_i)}{\lambda(F_i)} = 0, \tag{1.1}$$

where Δ denotes the symmetric difference and λ is a left invariant Haar measure on G.

Now, if Γ is a discrete countable amenable group and $\Gamma \curvearrowright (X, \mu, \mathcal{B})$ is an action of Γ by automorphisms on the standard probability space, we can define entropy of the action by

$$h_\mu(\Gamma \curvearrowright (X, \mu)) = \sup_{\mathcal{P}} \lim_{i\to\infty} \frac{1}{|F_i|} H_\mu \left(\bigvee_{\gamma \in F_i} \gamma^{-1} \mathcal{P} \right) \tag{1.2}$$

where the supremum is taken over all partitions with finite Shannon entropy. This definition does not depend on the choice of Følner sequence. Moreover, if $\mathcal{P}$ is a generating partition, i.e. the smallest Γ invariant σ algebra (up to measure zero) containing $\mathcal{P}$ is $\mathcal{B}$, then the supremum is achieved by it,

$$h_\mu(\Gamma \curvearrowright (X, \mu)) = \lim_{i\to\infty} \frac{1}{|F_i|} H_\mu \left(\bigvee_{\gamma \in F_i} \gamma^{-1} \mathcal{P} \right).$$

Since many arguments in ergodic theory involve averaging, the presence of Følner sets make amenable groups a fertile ground for the development of the theory.

Capturing this, Ornstein and Weiss' seminal work [OW87] extended many basic results, including Rokhlin's theorem, Shannon-Mcmillan theorem, and entropy theory, to locally compact, amenable, unimodular groups. Let us mention that expanding from a discrete countable setting to continuous groups presents several challenges. For example, defining entropy by using an analogous definition as (1.2), most atoms of $\bigvee_{\gamma\in F}\gamma^{-1}\mathcal{P}$ will typically have zero measure. This problem manifests itself, even in the case of the real line $\mathbb{R}$, and is generally avoided by defining the entropy of an $\mathbb{R}$ action as the entropy of the action restricted to its cocompact subgroup of integers.

J. Feldman [F70], found a way to define, directly, the entropy for actions of the continuous group $\mathbb{R}^n$. His approach is as follows: for a measure preserving action $\mathbb{R}\curvearrowright(X,\mu)$ and a finite partition $\mathcal{P}$, a subset $B\subset X$ is said to be a $(\mathcal{P},\varepsilon,K)$ ball if for every $x,y\in B$,

$$\frac{1}{K}\left|0<t<K:\mathcal{P}(tx)\neq\mathcal{P}(ty)\right|<\varepsilon,$$

where $\mathcal{P}(tx)\neq\mathcal{P}(ty)$ means tx and ty are in different atoms of the partition. Let $N(\mathcal{P},\varepsilon,K)$ denote the minimum number of $(\mathcal{P},\varepsilon,K)$ balls required to cover X, then

$$\lim_{\varepsilon\to 0}\limsup_{K\to\infty}\frac{1}{K}\log N(\mathcal{P},\varepsilon,K)$$

defines the entropy of the action.

Ornstein and Weiss [OW87] extended this idea to the amenable setting, but, because of noncommutativity, it warrants a detailed, careful analysis. Furthermore, they raised the question "What happens for nonamenable groups?" and gave an example which suggested that there might not be reasonable way to extend entropy theory for nonamenable groups. A basic property of the entropy for amenable groups is that it is non-increasing under factor maps. Recall that given two measure preserving actions $\Gamma\curvearrowright(X,\mu)$ and $\Gamma\curvearrowright(Y,\nu)$, a map $\Phi:(X,\mu)\to(Y,\nu)$ is a factor map if $\Phi_*\mu=\nu$ and it intertwines the action i.e. $\Phi(\gamma x)=\gamma\Phi(x)$ for every $\gamma\in\Gamma$ and almost every x. So, if Γ is amenable, then the Bernoulli shift with the base space $(\{0,1\},\kappa_2)$, which has entropy $\log 2$, cannot factor onto the Bernoulli shift with the base space $(\{0,1,2,3\},\kappa_4)$, which has entropy $\log 4$. In contrast, if the group $\Gamma=<a,b>$ is a non abelian rank two free group, a nonamenable group, then $\Phi:\mathbb{Z}_2^\Gamma\to(\mathbb{Z}_2\times\mathbb{Z}_2)^\Gamma$ defined by

$$\phi(x)_\gamma=(x_\gamma+x_{\gamma a},x_\gamma+x_{\gamma b})$$

gives a factor map.

Lewis Bowen [Bow10b] introduced entropy theory for the actions of countable sofic groups, providing a family of isomorphism invariants. Countable sofic groups were defined implicitly by Gromov [Gr], and explicitly by Weiss [Wei00], as a common generalization of amenable groups and residually finite groups. Several conjectures, including Gottshalk's surjunctivity conjecture [Gr] [Wei00], Connes' embedding conjecture [ES05], the determinant conjecture [ES05], and Kaplansky's direct finiteness conjecture [ES04], have been shown to be true for them. There are several definitions of soficity for countable groups; the following is best suitable for developing entropy theory;

Definition 1. Let Γ be a countable group. For $d\in\mathbb{N}$, let $\mathrm{Sym}(d)$ be a group of permutations on the set $\{1,2,\ldots,d\}$. We say that Γ is **sofic** if there is a sequence $\{d\}_{i=1}^\infty$ of positive integers and a sequence of maps $\{\sigma_i\}_{i=1}^\infty$ from Γ to $\mathrm{Sym}(d_i)$ such that they are asymptotically multiplicative

$$\lim_{i\to\infty}\frac{1}{d_i}\left|p\in\{1,2\ldots,d_i\}:\sigma_i(\gamma_1)\sigma_i(\gamma_2)p=\sigma_i(\gamma_1\gamma_2)p\right|=1$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and asymptotically free

$$\lim_{i\to\infty} \frac{1}{d_i} \left| p \in \{1, 2 \ldots, d_i\} : \sigma_i(\gamma_1)p \neq \sigma_i(\gamma_2)p \right| = 1$$

for all distinct $\gamma_1, \gamma_2 \in \Gamma$. The sequence of maps $\Sigma = \{\sigma_i : \Gamma \to \mathrm{Sym}(d_i)\}$ is called a **sofic approximation** to Γ.

For more information on countable sofic groups, see the survey by Pestov [Pes08]. Furthermore, it is unknown whether all countable groups are sofic.

Soficity gives the existence of approximate actions on finite spaces, which Bowen [Bow10b] exploited to define entropy for actions of countable sofic groups by counting finitary dynamic models that capture the action up to some error. His approach used partitions as the observational scale for the action and showed that the entropy is invariant over generating partitions with finite Shannon entropy. Moreover, as in the case of amenable groups, the entropy of Bernoulli shift action of countable sofic group is the same as the Shannon entropy of the base space.

This definition was well defined only in the presence of a generating partition with finite Shannon entropy. Using an operator algebraic framework, Kerr and Li [KL11] provided an alternative approach, thus enabling them to extend the entropy in the absence of a generating partition with finite Shannon entropy. They also defined the topological entropy for the continuous actions of countable sofic groups and established the variational principle connecting measure entropy and topological entropy.

The idea of soficity has also been extended to measured equivalence relations [EL10], discrete measured groupoids [DKP11], and unimodular random rooted networks [AL07]. Moreover, entropy theory has also been developed for class bijective extensions of sofic measured groupoids [Bow14].

In this thesis, we extend the notion of soficity from countable groups to locally compact groups and develop entropy theory for the actions of such groups. The motivation for our definition of a locally compact sofic group comes primarily from the following: suppose $\Gamma < G$ is a residually finite lattice in a locally compact group G. Then, if $\{\Gamma_i\}_{i=1}^{\infty}$ is a decreasing sequence of finite index normal subgroups of Γ, the finite measure homogenous spaces $\{G/\Gamma_i\}_{i=1}^{\infty}$ are the analogues of finite spaces $\{1, 2, \cdots, d_i\}$ from the discrete setting and will give a sofic approximation to the group G.

Definition 2 (Sofic Group)**.** Suppose G is a locally compact second countable group with a left invariant Haar measure λ. Let U be a precompact neighborhood of the identity e in G and $\epsilon > 0$. A finite measure space (M, vol) with a map $L : G \times M \to M$ is a (U, ϵ)**-sofic approximation** to G if there exist a measurable set $M_0 \subset M$ with $\mathrm{vol}(M_0) > (1 - \epsilon)\,\mathrm{vol}(M)$ satisfying the following for every $p \in M_0$:

1. the restriction of L to $U \times \{p\}$ is a measure space isomorphism onto its image;

2. $L(e, p) = p$;

3. for every $g, h \in U$ with $gh \in U$,

$$L(g, L(h, p)) = L(gh, p).$$

A **sofic approximation** to a group G is a sequence $\{(M_i, \mathrm{vol}_i, L_i)\}_{i=1}^{\infty}$ where, given any precompact neighborhood U of the identity and any $\varepsilon > 0$, there exists an i_0 such that for every $i > i_0$, the finite measure space $(M_i, \mathrm{vol}_i, L_i)$ is a (U, ε)-sofic approximation to G.

A group G is **sofic** if it admits a sofic approximation.

Remark 1. The measure space isomorphism in condition (1) depends on a chosen Haar measure and, therefore, the sofic approximation as well. Since the Haar measure is unique up to scaling, any sofic approximation with respect to one Haar measure gives a sofic approximation with respect to another Haar measure by rescaling the volume on the approximation space. Hence, the notion of a sofic group is independent of a Haar measure.

We show that all unimodular, amenable, locally compact groups are sofic. Moreover, if a co-finite subgroup H of a locally compact group G is sofic, then G itself is also sofic. It follows from [Mal40] that every countable linear group is residually finite, hence sofic, and therefore any locally compact group containing a countable linear group as a lattice is sofic.

For defining entropy, there are several approaches stemming from which viewpoint one witnesses the dynamics of the action as well as the scale of observation. For example, in Kolmogrov - Sinai entropy, one studies the dynamics on the space itself using a partition $\mathcal{P}$ as the scale. Kerr - Li [KL11], on the other hand, devised an approach in which they looked at the dynamics of the action on L^∞ space using functions as their scale of observation. This latter approach has an advantage in that functions allow one to witness the dynamical behavior at a much finer scale than the partition.

Using a similar approach, in chapter 2, we present a definition of entropy of a measure preserving action of a locally compact sofic group in an operator-algebraic framework, i.e. studying the dynamics on L^∞ space. This approach is based on counting "good models" for the action using sofic approximation. However, the term "good models" can be given several meanings, which lead to different definitions of the entropy. The main result of this chapter is that these differences are irrelevant and all approaches are equivalent, thereby, providing a tractable way to calculate the entropy.

Nonetheless, it is much simpler to express the entropy in terms of dynamics on the space itself rather than on L^{∞}, but it requires a topological structure on the space. That is, we need to assume that the group is acting continuously on a compact metric space. But, such compact models always exist for locally compact second countable groups, thus, there is no loss of generality if we restrict ourselves to this setting. In chapter 2 we also develop this spatial definition and show that it agrees with the operator algebraic approach.

Similar to the entropy of measure preserving actions, as an invariant for topological dynamical systems, topological entropy was introduced by Adler, Konheim and Mcandrew [AKM65]. In chapter 3 we introduce topological entropy for a continuous action of a locally compact sofic group on a compact metric space and establish the variational principle relating the measure entropy to the topological entropy:

Theorem 1.0.2 (Variational Principle)**.** *Suppose $\alpha : G \times X \to X$ is a continuous action of a locally compact sofic group G on a compact metric space X. For every sofic approximation Σ we have*

$$h_{\Sigma}^{top}(\alpha) = \sup_{\mu \in \mathscr{M}_G(X)} h_{\Sigma}^{meas}(\mu, \alpha),$$

where $\mathscr{M}_G(X)$ is the space of G-invariant Borel probability measures on X.

The natural generalization of the Bernoulli shift action of countable groups to continuous groups are Poisson point processes. On a topological group, the Poisson point process can be defined as a process on the space of discrete subsets of the group and can be viewed as a random scattering of points, where the probability of seeing a point in some set in the group is dependent only on the Haar measure of the set and is independent for disjoint sets. This forms a probability space on which the group acts by measure preserving translations. If Γ is countable then Poisson point processes over Γ can directly seen to be measurably conjugate to Bernoulli shifts. There are several Poisson point processes of different intensities on the group, where the intensity specifies the expected number of points in a set of measure one of a group. In chapter 4, we study the entropy of Poisson processes over a sofic group and show the following:

Theorem 1.0.3. *Suppose G is a nondiscrete locally compact sofic group with Haar measure λ and $(X, \mathbb{P}_\kappa)$ is a Poisson point process of intensity $\kappa > 0$ on G. Then for any sofic approximation Σ to G we have*

$$h_\Sigma^{meas}(G \curvearrowright (X, \mathbb{P}_\kappa)) = \infty.$$

In chapter 5, we study the relationship between the entropies of an action of a locally compact group and the restriction of the action to a lattice in the group. For the actions of integers it is well known that if $T : (X, \mu) \to (X, \mu)$ is an automorphism, then

$$h_\mu(T^n) = |n|\, h_\mu(T)$$

for every $n \in \mathbb{Z} \backslash \{0\}$. Also, if $\Lambda < \Gamma$ is a finite index subgroup in a countable amenable group Γ and $\Gamma \curvearrowright (X, \mu)$ is a measure preserving action, we have

$$h_\mu(\Lambda \curvearrowright (X, \mu)) = [\Lambda : \Gamma] h_\mu(\Gamma \curvearrowright (X, \mu)).$$

For further information, see [Dan01]. Extending these results, we show the following:

Theorem 1.0.4. *Let $\Gamma < G$ be a lattice in a locally compact group G with a Haar measure λ. Suppose Σ is a sofic approximation to Γ and $\widetilde{\Sigma}$ be the induced sofic approximation to G. Let $\alpha : G \times (X, \mu) \to (X, \mu)$ be a continuous measure preserving* *action on a compact metric space X and $\alpha_\Gamma : \Gamma \times (X, \mu) \to (X, \mu)$ be its restriction. Then,*

$$h^{meas}(\alpha_\Gamma, \Sigma) \leq \lambda(G/\Gamma) h^{meas}(\alpha, \widetilde{\Sigma}), \quad \textit{and} \quad h^{top}(\alpha_\Gamma, \Sigma) = \lambda(G/\Gamma) h^{top}(\alpha, \widetilde{\Sigma})$$

where $\lambda(G/\Gamma)$ is the covolume of the lattice Γ.

Chapter 2

Sofic Groups

Countable sofic groups have many different definitions and characterizations. One such definition was given in the Introduction (see Def 1) using maps from the group into symmetric groups. It follows from this definition that a countable group Γ is sofic if and only if every finitely generated subgroup of Γ is sofic. Although the definition using maps lends itself well to define entropy; a more intuitive characterization of soficity can be given using finite graphs.

In order to present it, we will need the notion of Cayley graphs. Let Γ be a finitely generated group and $S \subset \Gamma$ a finite symmetric (i.e. $S = S^{-1}$) generating set. The Cayley graph of Γ is a directed graph $\mathscr{G}(\Gamma, S)$ with the vertex set Γ and edges labelled by the elements of S; for every $s \in S$, there is a s-labeled directed edge $(\gamma, s\gamma)$ for all $\gamma \in \Gamma$. Let $B_r(e)$ denote the ball of radius r around the identity e in $\mathscr{G}(\Gamma, S)$. Then from [Wei00], we have the following characterization of soficity:

Definition 3. A finitely generated group Γ is **sofic** if there exists a finite symmetric generating set S such that for any $r \in \mathbb{N}$ and any $\varepsilon > 0$ there is a finite directed graph (V, E) with edges labeled by S, and a subset $V_0 \subset V$ satisfying:

1. $|V_0| \geq (1 - \varepsilon) |V|$,

2. for each $v \in V_0$ there is a labeled graph isomorphism $\phi_v : B_r(e) \to B_r(v)$,

where $B_r(v)$ is the radius r ball around v in the graph (V, E).

The equivalence of the two characterizations Def 1 and Def 3 was proven in [ES04], which in particular also shows that it is independent of generating set. We mention that the class of countable sofic groups contains all amenable groups and residually finite groups. Moreover it is closed under taking subgroups, direct products, inverse limits, direct limit and free products with amalgamation over amenable subgroups.

Building on the above characterization, we extend the idea of soficity beyond countable groups. Throughout the thesis we will assume the group G is a **locally compact, second countable, topological group**. Although just assuming σ -compactness and Hausdorff will suffice for many arguments as the σ-compactness assures the needed accessibility to large scales using compact sets and Hausdorff ensures that arbitrarily small scales can be seen by open sets.

Definition 4 (Sofic Group)**.** Suppose G is a locally compact second countable group with a left Haar measure λ. Let U be a precompact neighborhood of the identity e in G and $\epsilon > 0$. A finite measure space (M, vol) with a measurable map $L : G \times M \to M$ is a (U, ϵ)**-sofic approximation** to G if there exists a measurable set $M_0 \subset M$ with $\mathrm{vol}(M_0) > (1-\epsilon)\,\mathrm{vol}(M)$ satisfying the following for every $p \in M_0$:

1. the restriction of L to $U \times \{p\}$ is a measure space isomorphism onto its image;

2. $L(e, p) = p$;

3. for every $g, h \in U$ with $gh \in U$,

$$L(g, L(h, p)) = L(gh, p).$$

A **sofic approximation** to G is a sequence $\{(M_i, \mathrm{vol}_i, L_i)\}_{i=1}^{\infty}$ if given any precompact neighborhood U of the identity and any $\varepsilon > 0$ there exist i_0 such that for every $i > i_0$, the finite measure space $(M_i, \mathrm{vol}_i, L_i)$ is a (U, ε)-sofic approximation to G.

The group G is **sofic** if it admits a sofic approximation.

A sofic approximation to a group G depends on the chosen Haar measure λ as an isomorphism in condition (1) is with respect to the measure λ on U. If we choose another Haar measure λ' which will just be a scalar multiple of λ, then scaling the measure vol accordingly on M ensures that we have a sofic approximation to G with respect to λ'

We check that for a discrete countable group this notion of soficity agrees with the previous one.

Proposition 2.0.1. *Let Γ be a discrete countable group, then the two characterizations Def 3 and Def 4 of soficity for Γ are equivalent.*

Proof. ($3 \Rightarrow 4$) Let $|\cdot|$ denote the counting measure on Γ, thereby our choice for a Haar measure. Given a finite set $F \subset \Gamma$ and a positive real $\varepsilon > 0$, we need to furnish a (F, ε)-sofic approximation to Γ. So, let $\Gamma_F < \Gamma$ be a subgroup generated by F. By hypothesis we have a finite graph (V, E) with edges labeled by F, which has a subset $V_0 \subset V$ such that $|V_0| \geq (1 - \varepsilon)|V|$ and for each $v \in V_0$ there is a labeled graph isomorphism $\phi_v : B_2(e) \to B_2(v)$ where $B_2(e)$ is a $2-$neighborhood in the Cayley graph $\mathscr{G}(\Gamma_F, F)$. Define a map $L : F \times V \to V$ by

$$L(f, v) = \phi_v(f)$$

for every $v \in V_0$, $f \in F$ and $L(f, v) = v$ for every $v \in V \backslash V_0$. Since the $2-$neighborhood in V is isomorphic to that of the Cayley graph with generating set F, it is clear that for every $v \in V_0$ and $f, f' \in F$, we have $L(ff', v) = L(f, L(f', v))$. Also, the map $L(\,\cdot\,, v) : F \to V$ is injective whenever $v \in V_0$ and therefore (V, L) with the counting measure is a (F, ε) sofic approximation to Γ.

(4⇒3) From the comments at the start of the chapter, it suffices to show the implication for a finitely generated group Γ. Let $\mathscr{G}(\Gamma, S)$ be a Cayley graph of Γ. Suppose a natural number $r \in \mathbb{N}$ and a positive real $\varepsilon > 0$ is given. By assumption there exists a measure space (M, vol) together with a map $L : B_r(e) \times M \to M$ which is a $(B_r(e), \varepsilon)$ sofic approximation to Γ.

Therefore for every $p \in M_0$, the image $L(B_r(e), p)$ is a discrete set with $|B_r(e)|$ many elements. Define a graph (V, E) with the vertex set

$$V = \bigcup_{p \in M_0} L(p, B_r(e))$$

and for every $p \in V$ and $s \in S$ let there be a $s-$labelled edge $(p, L(s, p))$. Set $V_0 = M_0$. As $L(e, p) = p$, we have

$$|V_0| = \mathrm{vol}(M_0) \geq (1 - \varepsilon)\,\mathrm{vol}(M) \geq (1 - \varepsilon)\,|V|\,.$$

As for every $s, t \in S$ and $p \in V_0$, $L(st, p) = L(s, L(t, p))$, we conclude that the $r-$neighborhood in the graph (V, E) of a vertex $v \in V_0$ is graph isomorphic to $B_r(e)$. □

Proposition 2.0.2. *Every unimodular locally compact second countable amenable group G is sofic.*

Proof. To show this, the idea is that given a compact set K in the group, we can find a Følner set F in the group for which a K neighborhood Kf of most points $f \in F$

lives inside F. More precisely, fix λ to be a bi-invariant Haar measure on G, and suppose a positive real $\varepsilon > 0$ and $U \subset G$, a precompact neighborhood of the identity are given. Our aim is to find a (U, ε) sofic approximation to G.

Set $K = \bar{U}$ to be the closure of U. Since G is amenable, using condition (1.1), we get a set $F \subset G$ with $\lambda(KF \Delta F) \leq \varepsilon \lambda(F)$, and therefore

$$\lambda(F) \geq \frac{\lambda(KF)}{(1+\varepsilon)} \geq (1-\varepsilon)\lambda(KF).$$

Given an element $u \in U$ and $f \in KF$, define a map $L : U \times KF \to KF$ as

$$L(u, f) = \begin{cases} uf & \text{if } f \in F \\ f & \text{if } f \notin F \end{cases}$$

For every $f \in F$, the map $L(\,\cdot\,, f) : U \to KF$ is the right translation by f and therefore a measure space isomorphism onto its image. Thus we conclude that (KF, λ, L) is a (U, ε)-sofic approximation to G. $\square$

We now turn our attention to the groups containing sofic subgroups. Suppose G is a locally compact group with a left invariant Haar measure λ. For each $g \in G$, the measure λ_g defined by $\lambda_g(A) = \lambda(Ag)$ is also a left invariant Haar measure on G, and hence there is $\Delta(g) \in \mathbb{R}^+$ such that $\lambda_g = \Delta(g)\lambda$. The function $\Delta : G \to \mathbb{R}^+$ is called the **modular function** of G and G is unimodular if and only if $\Delta \equiv 1$.

Now, suppose $H < G$ is a closed subgroup and denote by G/H the space of left cosets of H in G. There is a natural action of G on the coset space G/H by left translations and there exists a measure $\lambda_{G/H}$ on G/H invariant under this action if and only if $\Delta_G(h) = \Delta_H(h)$ for every $h \in H$. Moreover, if such a measure exists it is unique up to scalar multiples and suitably normalized satifies for all $f \in C_c(G)$:

$$\int_G f(g)\, d\lambda_G(g) = \int_{G/H} \int_H f(gh)\, d\lambda_H(h) d\lambda_{G/H}(gH), \tag{2.1}$$

where λ_G and λ_H are left invariant Haar measures on G and H respectively.

If G/H admits a finite invariant measure, then H is said to be a **cofinite subgroup** of G. A cofinite discrete subgroup $\Gamma < G$ is called a **lattice** in G. Our next proposition allows us to conclude soficity for a group if it contains a sofic cofinite subgroup. The assumption of second countability will be crucial for us as it provides the existence of a Borel section for the the canonical projection $\pi : G \to G/H$.

Proposition 2.0.3. *Suppose G is a unimodular, locally compact, second countable group with a Haar measure λ_G. Let $H \leq G$ be a cofinite, unimodular subgroup of G with a Haar measure λ_H. If $\Sigma = \{(M_i, \mathrm{vol}_i, \sigma_i)\}_{i=1}^{\infty}$ is a sofic approximation to H, then there exists an induced sofic approximation $\widetilde{\Sigma} = \{(G/H \times M_i, \lambda_{G/H} \times \mathrm{vol}_i, L_i)\}_{i=1}^{\infty}$ to G.*

Proof. The key idea is that of Mackey's induced action which allows one to induce an action of G from an action of H. Let $\pi : G \to G/H$ be the canonical projection, then from [Mac52](lemma 1.1) there is a Borel cross section $\theta : G/H \to G$ of π with $\theta(\pi(e)) = e$, where e is the identity element of G. Moreover, for every compact set $K \subset G$, the set $\theta(\pi(K)) \subset G$ has a compact closure.

Given $g \in G$ and $x \in G/H$ define a cocycle $\alpha : G \times G/H \to H$ by

$$\alpha(g, x) = \theta(gx)^{-1}\, g\, \theta(x).$$

Since θ is a cross section, we see that $\alpha(g, x) \in H$ and definition of α implies

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 x)\alpha(g_2, x)$$

whenever $g_1, g_2 \in G$ and $x \in X$.

Suppose $U \subset G$, a precompact neighborhood of the identity and $\varepsilon > 0$, a real number are given. Our first task is to show that there is a compact set $X_0 \subset G/H$ of large measure such that the image of α restricted to the set $U \times X_0$ is contained in a compact subset of H. Let $[e] = \pi(e) \in G/H$ be the identity coset. Observe that for every $g \in U$,

$$\alpha(g, [e]) = \theta(g[e])^{-1} g\theta([e]) = \theta(\pi(g))^{-1} g$$

is contained in the compact set

$$\theta(\pi(\bar{U}))^{-1}\, \bar{U}.$$

And for elements other than $[e]$ of G/H, we will use the cocycle identity to reduce them to $[e]$. Let $X_0 \subset G/H$ be a compact set with its measure $\lambda_{G/H}(X_0) \geq (1 - \varepsilon')\lambda_{G/H}(G/H)$. Since, π is an open continuous map, there is a compact subset $K \subset G$ with $\pi(K) = X_0$. For every $x_0 \in X_0$, let $g_0 := \theta(x_0)$ and observe that,

$$\alpha(g, x_0) = \alpha(gg_0 g_0^{-1}, x_0) = \alpha(gg_0, g_0^{-1} x_0)\alpha(g_0^{-1}, x_0) = \alpha(gg_0, [e])e = \alpha(gg_0, [e]).$$

Therefore for every $g \in U$ and $x_0 \in X_0$, we see that $g_0 \in \tilde{K} = \theta \circ \pi(K)$,

$$\alpha(g, x_0) = \alpha(gg_0, [e]) = \theta(\pi(gg_0))^{-1} gg_0 \subset \theta(\pi(\bar{U}\tilde{K}))^{-1} \bar{U}\tilde{K}$$

is contained in a compact subset of H.

Now, let $(M, \mathrm{vol}, \sigma)$ be a (U_H, ε')-sofic approximation to the group H such that U_H contains the image of α restricted to $U \times X_0$. Given $g \in U$, $x \in G/H$ and $p \in M$, define a map $L : U \times (G/H \times M) \to (G/H \times M)$ by

$$L(g, (x, p)) = \begin{cases} (gx, \sigma(\alpha(g, x), p)) & \text{if } x \in X_0 \\ (gx, p) & \text{if } x \notin X_0 \end{cases}$$

Equipping the set $G/H \times M$ with the product measure $\lambda_{G/H} \times \mathrm{vol}$, we show that L gives us a (U, ε)-sofic approximation to G.

Firstly, if $M_0 \subset M$ is the good set the sofic approximation $(M, \mathrm{vol}, \sigma)$ (this means that σ restricted to $U_H \times \{p\}$ is an isomorphism onto its image for every $p \in M_0$), it is immediate from the cocycle identity of α that

$$L(g_1 g_2, (x, p)) = L(g_1, L(g_2, (x, p))),$$

whenever $g_1, g_2 \in U$ with $g_1 g_2 \in U$ and $(x, p) \in X_0 \times M_0$.

Secondly, we have to show that L satisfies the local isomorphism condition. Fix $x_0 \in X_0$ and $p \in M_0$. Since σ restricted to $U_H \times \{p\}$ is a local isomorphism, for notational convenience, whenever $h \in U_H$, we identify $\sigma(h, p)p$ with h. With this identification we can write the map

$$T := L(\,\cdot\,, (x_0, p)) : U \to G/H \times U_H$$

by

$$T(g) = L(g, (x_0, p)) = (gx_0, \alpha(g, x_0)).$$

The map T is injective with its inverse given by the map

$$G/H \times U_H \ni (x,h) \to \theta(x)h\theta(x_0)^{-1}.$$

Now, let $B \subset (G/H \times U_H)$ be a measurable subset. Since λ_G is right invariant, and using the formula (2.1) we see that,

$$\begin{aligned}\int_G \chi_B \circ T(g)\, d\lambda_G(g) &= \int_G \chi_B \circ T(g\theta(x_0)^{-1})\, d\lambda_G(g) \\ &= \int_{G/H}\int_H \chi_B \circ T(\theta(x)h\theta(x_0)^{-1})\, d\lambda_H(h)\, d\lambda_{G/H}(x).\end{aligned}$$

The definition of T and α leads to

$$T(\theta(x)h\theta(x_0)^{-1}) = (x,h)$$

and we thus have

$$\int_G \chi_B \circ T(g)\, d\lambda_G(g) = \int_{G/H}\int_H \chi_B(x,h)\, d\lambda_H(h)\, d\lambda_{G/H}(x).$$

Therefore, T is a measure isomorphism onto its image. If $\varepsilon' \leq 1-(1-\varepsilon)^{1/2}$, then

$$(\lambda_{G/H} \times \mathrm{vol})(X_0 \times M_0) \geq (1-\varepsilon').(\lambda_{G/H} \times \mathrm{vol})(G/H \times M).$$

and we conclude that $(G/H \times M, \lambda_{G/H} \times \mathrm{vol}, L)$ is a (U,ε)- sofic approximation to G. $\square$

Chapter 3

Measure Entropy

In this chapter we introduce the entropy for measure preserving actions of a locally compact sofic group. In particular, our approach is set up in an operator algebraic framework and we show that the defined notion of entropy is an invariant of the action.

3.1 Terminology

We first set up some terminology and notation that will be used in the thesis. We will consider commutative von Neumann algebras of the form $L^\infty(X,\mu)$ where (X,μ) is a standard probability space. A **projection** in an algebra is an element satisfying $p = p^2$ and $p = p^*$; projections in $L^\infty(X,\mu)$ correspond to measurable subsets of X up to measure zero.

A linear map $\phi : L^\infty(X,\mu) \to L^\infty(Y,\nu)$ is said to be **unital** if $\phi(1) = 1$ and **positive** if $\phi(f) \geq 0$ whenever $f \geq 0$. Also, a unital linear map between commutative von Neumann algebras is said to be a **homomorphism** if $\phi(f_1 f_2) = \phi(f_1)\phi(f_2)$ for every $f_1, f_2 \in L^\infty(X,\mu)$. Moreover, a unital homomorphism is automatically positive.

Throughout this section we assume that G is a locally compact sofic group with an action $\alpha : G \times (X, \mu) \to (X, \mu)$ by measure space automorphisms. Then, we have an action of G on $L^\infty(X, \mu)$ by automorphisms, which we will also denote by α, that is for every $f \in L^\infty(X, \mu)$ and $g \in G$ the function $\alpha_g(f)$ is defined by $\alpha_g(f)(x) := f(g^{-1}x)$ where $x \in X$.

A set $\mathcal{S} \subset L^\infty(X, \mu)$ is said to be **dynamically generating** if it is not contained in any proper G-invariant von Neumann sub-algebra of $L^\infty(X, \mu)$. Note that, by Kaplansky's density theorem, von Neumann subalgebras of $L^\infty(X, \mu)$ correspond to unital *-subalgebras of $L^\infty(X, \mu)$ which are closed in L^2 norm.

3.2 Entropy via Unital Homomorphisms

Let $\mathcal{P} = \{p_n\}_{n=1}^\infty$ be a dynamically generating sequence in the unit ball(with respect to L^∞ norm) of $L^\infty(X, \mu)$. If $\mathcal{P}$ is a partition of unity consisting of projections then it corresponds to the usual notion of a generating partition. We will need the following lemma:

Lemma 3.2.1. *There exists a countable set $\Gamma \subset G$ such that the minimal Γ-invariant von Neumann subalgebra of $L^\infty(X, \mu)$ containing $\mathcal{P}$ is $L^\infty(X, \mu)$.*

Proof. If $\Gamma \subset G$ is a countable dense subset, by Lemma 3.3.1 the Γ-invariant von Neumann subalgebra of $L^\infty(X, \mu)$ containing $\mathcal{P}$ is also G-invariant and hence all of $L^\infty(X, \mu)$. $\square$

Let Γ be as in the above lemma, then, given a finite set $E \subset \Gamma$ and $n \in \mathbb{N}$, let $\mathcal{P}_{E,n} \subset L^\infty(X, \mu)$ be the set of functions of the form

$$\prod_{i=1}^{j} \alpha_{\gamma_i}(p_i)$$

for every $1 \leq j \leq n$ and $\gamma_1, \ldots, \gamma_j \in E$. In the following definition, we consider L^2 bounded homomorphisms from $L^\infty(X, \mu)$ into a sofic approximation space $L^\infty(M, \mathrm{vol})$ that are approximately measure preserving and equivariant at the scale seen through $\mathcal{P}$. These will serve as our "good models" to capture the dynamics of the action.

Definition 5. Let $n \in \mathbb{N}$, $\delta > 0$, $E \subset \Gamma$ be a finite subset and $U \subset G$ a precompact neighborhood of the identity. Suppose (M, vol, L) is a (U, β)-sofic approximation to G and let $\overline{\mathrm{vol}}$ denote the probability measure obtained by normalizing vol so that $\overline{\mathrm{vol}}(M) = 1$. Given $C \geq 1$, define $\mathrm{Hom}_C(\mathcal{P}, \Gamma, U, E, n, \delta, L)$ to be the set of all unital homomorphisms

$$\phi : L^\infty(X, \mu) \to L^\infty(M, \overline{\mathrm{vol}})$$

satisfying:

1. $\|\phi(f)\|_2 \leq C \|f\|_2$ for all $f \in L^\infty(X, \mu)$,
2. $\left|\int \phi(f) d\,\overline{\mathrm{vol}} - \int f d\mu\right| \leq \delta$ for every $f \in \mathcal{P}_{E,n}$,
3. $\|\phi \circ \alpha_g(f) - L_g \circ \phi(f)\|_2 \leq \delta$ for all $g \in U$ and $f \in \{\, p_1, p_2, \ldots, p_n \,\}$.

On the set of maps from $L^\infty(X, \mu)$ into $L^\infty(M, \overline{\mathrm{vol}})$ we define the pseudometric

$$\rho_{\mathcal{P}}(\phi, \psi) = \sum_{i=1}^{\infty} \frac{1}{2^i} \|\phi(p_i) - \psi(p_i)\| \,.$$

For any $\varepsilon \geq 0$ and a pseudometric space (Y, ρ) we will denote by $N_\varepsilon(Y, \rho)$ to be the maximal cardinality of an ε-separated subset of Y with respect to ρ.

Definition 6. Suppose $\Sigma = \{\, (M_i, \mathrm{vol}_i, L_i) \,\}_{i=1}^{\infty}$ is a sofic approximation sequence to G. Let $\mathcal{P}$ be a generating sequence in the unit ball of $L^\infty(X, \mu)$, $E \subset \Gamma$ a finite set,

$U \subset G$ a precompact neighborhood of the identity, $n \in \mathbb{N}$ and $\delta, \varepsilon > 0$. We define

$$\begin{aligned}
h^{\varepsilon}_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma, U, E, n, \delta) &= \limsup_{i \to \infty} \frac{1}{\mathrm{vol}_i(M_i)} \log N_{\varepsilon}\big(\mathrm{Hom}_C(\mathcal{P}, \Gamma, U, E, n, \delta, L_i), \rho_{\mathcal{P}}\big), \\
h^{\varepsilon}_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma, U, E, n) &= \inf_{\delta > 0} h^{\varepsilon}_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma, U, E, n, \delta), \\
h^{\varepsilon}_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma, U, E) &= \inf_{n \in \mathbb{N}} h^{\varepsilon}_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma, U, E, n), \\
h^{\varepsilon}_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma, U) &= \inf_{E} h^{\varepsilon}_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma, U, E), \\
h^{\varepsilon}_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma) &= \inf_{U} h^{\varepsilon}_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma, U), \\
h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma) &= \sup_{\varepsilon > 0} h^{\varepsilon}_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma)
\end{aligned}$$

where the infimum in E is over all finite subsets of Γ and the infimum in U is over all precompact neighborhoods of the identity.

Remark 2. The quantity $h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma)$ depends on the chosen sofic approximation Σ which we are suppressing in the notation.

Remark 3. We can assume that $p_1 = 1$. This is possible because if $\mathcal{P}'$ is the sequence with $p_1 = 1$ and the rest of p_i's shifted to one place right in the sequence, then, since we are considering unital homomorphisms, we have

$$\mathrm{Hom}_C(\mathcal{P}, \Gamma, U, E, n+1, \delta, L_i) \subset \mathrm{Hom}_C(\mathcal{P}', \Gamma, U, E, n, \delta, L_i) \subset \mathrm{Hom}_C(\mathcal{P}, \Gamma, U, E, n, \delta, L_i)$$

and hence $h_{\mathrm{Hom}_C}(\mathcal{P}', \Gamma) = h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma)$.

Our aim now is to show that the quantity $h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma)$ only depends on the G-invariant von Neumann subalgebra generated by $\mathcal{P}$ or in other words – it is independent of $\mathcal{P}$ and Γ. Moreover, later, we will show that $h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma)$ does not depend on the choice of $C \geq 1$.

Proposition 3.2.2. *Let Γ_1 and Γ_2 be countable subsets of G such that $\mathcal{P}$ is a dynamically generating sequence for both Γ_1 and Γ_2. Then*

$$h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma_1) = h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma_2).$$

Proof. The basic idea is that if we have an element ϕ serving as a "good model" for Γ_2, then it can serve as a "good model" with respect to Γ_1, albeit up to a different tolerance δ. The difference arises from the condition (2) in the definition of ϕ.

To carry out the argument, using Remark 3, we assume $p_1 = 1$. Let $E \subset \Gamma_1$ be a finite set, $U \subset G$ be a precompact neighborhood of the identity, $n \in \mathbb{N}$ and $\delta > 0$. Since Γ_2 is dynamically generating for $\mathcal{P}$, by Kaplansky's density theorem we can find a finite subset $E' \subset \Gamma_2$ and a natural number $n' > n$ such that any $f \in \mathcal{P}_{E,n}$ can be approximated by elements of the form

$$\tilde{f} = \sum_{f' \in \mathcal{P}_{E',n'}} c_{f,f'} f'$$

where, $c_{f,f'} \in \mathbb{R}$, and

$$\left\| f - \tilde{f} \right\|_2 \leq \frac{\delta}{2(C+1)}.$$

Set $\mathcal{M} = \max_{f \in \mathcal{P}_{E,n}} \max_{f' \in \mathcal{P}_{E',n'}} |c_{f,f'}|$ and $\delta' = \delta / 2\mathcal{M}|\mathcal{P}_{E',n'}|$. We will show

$$\mathrm{Hom}_C(\mathcal{P}, \Gamma_2, U, E', n', \delta', L) \subset \mathrm{Hom}_C(\mathcal{P}, \Gamma_1, U, E, n, \delta, L).$$

If $\phi \in \mathrm{Hom}_C(\mathcal{P}, \Gamma_2, U, E', n', \delta', L)$ we see that

$$\begin{aligned}\left|\int \phi(\tilde{f})\, d\overline{\mathrm{vol}} - \int \tilde{f} d\mu\right| &\leq \sum_{f' \in \mathcal{P}_{E',n'}} |c_{f,f'}| \left|\int \phi(f')\, d\overline{\mathrm{vol}} - \int f'\, d\mu\right| \\ &\leq \mathcal{M}|\mathcal{P}_{E',n'}|\delta' \leq \delta/2.\end{aligned}$$

And therefore using L^2 boundedness of ϕ, for every $f \in \mathcal{P}_{E,n}$, we thus have

$$\begin{aligned}\left|\int \phi(f)\, d\overline{\mathrm{vol}} - \int f\, d\mu\right| &\leq \left|\int \phi(f)\, d\overline{\mathrm{vol}} - \int \phi(\tilde{f})\, d\overline{\mathrm{vol}}\right| + \left|\int \phi(\tilde{f})\, d\overline{\mathrm{vol}} - \int \tilde{f} d\mu\right| \\ &\qquad + \left|\int \tilde{f} d\mu - \int f d\mu\right| \\ &\leq \left\|\phi(f) - \phi(\tilde{f})\right\|_2 + \frac{\delta}{2} + \left\|f - \tilde{f}\right\|_2 \\ &\leq (C+1)\left\|f - \tilde{f}\right\|_2 + \frac{\delta}{2} \leq \delta.\end{aligned}$$

This shows ϕ satisfies the δ-measure preserving condition. Since the other two conditions and the pseudometric $\rho_{\mathcal{P}}$ are independent of Γ_2, we conclude

$$h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma_2) \leq h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma_1).$$

Symmetrically the reverse inequality holds and thus $h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma_2) = h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma_1)$. $\square$

We now show that the defined notion of entropy $h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma)$ is independent of the chosen generating sequence.

Theorem 3.2.3. *Let $\alpha : G \times (X, \mu) \to (X, \mu)$ be a measure preserving action of a sofic group G. Suppose $\mathcal{P} = \{\, p_i \,\}_{i=1}^{\infty}$ and $\mathcal{Q} = \{\, q_i \,\}_{i=1}^{\infty}$ are dynamically generating sequences in the unit ball of L^∞ for the action α and $\Gamma \subset G$ is a countable set that is* *generating for both $\mathcal{P}$ and $\mathcal{Q}$. Then,*

$$h_{\mathrm{Hom}_C}(\mathcal{P},\Gamma) = h_{\mathrm{Hom}_C}(\mathcal{Q},\Gamma).$$

Proof. We split the proof into two parts – first we show that every good enough model ϕ with respect to $\mathcal{Q}$ can serve as a model with respect to $\mathcal{P}$, and secondly that the separation of models with respect to $\rho_{\mathcal{Q}}$ guarantees some level of separation with respect to $\rho_{\mathcal{P}}$. We assume $p_1 = 1$ and $q_1 = 1$.

Part 1:

Let $U \subset G$ be a precompact neighborhood of the identity, $E \subset \Gamma$ be a finite set, $n \in \mathbb{N}$, and $\delta, \varepsilon > 0$. Kaplansky's density theorem gives a finite set $E' \subset \Gamma$ and $n' \in \mathbb{N}$ such that every $f \in \mathcal{P}_{E,n}$ can be approximated by

$$\tilde{f} = \sum_{f' \in \mathcal{Q}_{E',n'}} c_{f,f'} f',$$

where $c_{f,f'} \in \mathbb{R}$ and $\left\| f - \tilde{f} \right\|_2 \leq \delta/4C$. Set $\mathcal{M} = \max_{f \in \mathcal{P}_{E,n}} \max_{f' \in \mathcal{Q}_{E',n'}} |c_{f,f'}|$.

Let $U' \subset G$ be a neighborhood of the identity with $UE' \subset U'$ and $\delta' = \delta/9\mathcal{M}n' \left|\mathcal{Q}_{E',n'}\right|$. We will show that if (M, vol, L) is a good enough sofic approximation, by which we mean that it is a (V,β)-sofic approximation with V large enough and β small enough, then we have the inclusion

$$\mathrm{Hom}_C(\mathcal{Q},\Gamma,U',E',n',\delta',L) \subset \mathrm{Hom}_C(\mathcal{P},\Gamma,U,E,n,\delta,L).$$

So, let $\phi \in \mathrm{Hom}_C(\mathcal{Q},\Gamma,U',E',n',\delta',L)$. If $g \in U$ and $f' = \prod_{k=1}^{n'} \alpha_{\gamma_k}(q_k) \in$

$\mathcal{Q}_{E',n'}$, $g \in U$, then ϕ being a homomorphism implies

$$\begin{aligned}
\|\phi \circ \alpha_g(f') - L_g \circ \phi(f')\|_2 &= \left\| \prod_{k=1}^{n'} \phi \circ \alpha_g \circ \alpha_{\gamma_k}(q_k) - \prod_{k=1}^{n'} L_g \circ \phi \circ \alpha_{\gamma_k}(q_k) \right\|_2 \\
&\leq \sum_{k=1}^{n'} \|\phi \circ \alpha_g \circ \alpha_{\gamma_k}(q_k) - L_g \circ \phi \circ \alpha_{\gamma_k}(q_k)\|_2 \\
&\leq \sum_{k=1}^{n'} \|\phi \circ \alpha_{g\gamma_k}(q_k) - L_{g\gamma_k} \circ \phi(q_k)\|_2 + \|(L_{g\gamma_k} - L_g \circ L_{\gamma_k})(\phi(q_k)\|_2 \\
&\qquad + \left\|L_g\big(L_{\gamma_k} \circ \phi(q_k) - \phi \circ \alpha_{\gamma_k}(q_k)\big)\right\|_2 .
\end{aligned}$$

To estimate the above three terms, observe that our choice of U' gives us the bounds

$$\|\phi \circ \alpha_{g\gamma_k}(q_k) - L_{g\gamma_k} \circ \phi(q_k)\| \leq \delta' \text{ and } \|\phi \circ \alpha_{\gamma_k}(q_k) - L_{\gamma_k} \circ \phi(q_k)\| \leq \delta'.$$

Moreover, if (M, vol, L) is a (V, β)-sofic approximation, for every function $F \in L^\infty(M, \overline{\mathrm{vol}})$, the local isomorphism condition of soficity translates to

$$\|L_g(F)\|_2 \leq \|F\|_2 + \beta \|F\|_\infty ,$$

whenever $g \in V$ and the cocycle property says

$$\|(L_{g_1 g_2} - L_{g_1} \circ L_{g_2})(F)\|_2 \leq 2\beta \|F\|_\infty ,$$

for every $g_1, g_2 \in V$ with $g_1 g_2 \in V$. Therefore, using the above inequalities, if

(M, vol, L) is a good enough sofic approximation, we thus have for every $f' \in \mathcal{P}_{E',n'}$,

$$\|\phi \circ \alpha_g(f') - L_g \circ \phi(f')\|_2 \leq \sum_{k=1}^{n'} 3\delta' = 3n'\delta'.$$

Given any $p \in \{ p_1, \dots, p_n \}$ and $g \in U$, we need to bound $\|\phi \circ \alpha_g(p) - L_g \circ \phi(p)\|_2$. However, any such p can be approximated by $\tilde{p}$, a linear combination of elements from the set $\mathcal{Q}_{E',n'}$. Therefore, for small enough β we see that

$$\begin{aligned}
&\|\phi \circ \alpha_g(p) - L_g \circ \phi(p)\|_2 \\
&\quad \leq \|\phi \circ \alpha_g(p) - \phi \circ a_g(\tilde{p})\|_2 + \|\phi \circ \alpha_g(\tilde{p}) - L_g \circ \phi(\tilde{p})\|_2 + \|L_g \circ \phi(\tilde{p}) - L_g \circ \phi(p)\|_2 \\
&\quad \leq C\|p - \tilde{p}\|_2 + \sum_{f' \in \mathcal{Q}_{E',n'}} |c_{p,f'}| \,.\, \|\phi \circ \alpha_g(f') - L_g \circ \phi(f')\|_2 + C\|p - \tilde{p}\|_2 + \beta \\
&\quad \leq 2C\|p - \tilde{p}\|_2 + 3\mathcal{M}n'\delta' |\mathcal{Q}_{E',n'}| + \beta \leq \delta/2 + \delta/3 + \beta \leq \delta
\end{aligned} \tag{3.1}$$

Now, suppose $f \in \mathcal{P}_{E,n}$ is given, and let $\tilde{f}$ be an approximation to it, then

$$\begin{aligned}
\left| \int \phi(\tilde{f})\, d\overline{\mathrm{vol}} - \int \tilde{f} d\mu \right| &\leq \sum_{f' \in \mathcal{Q}_{E',n'}} |c_{f,f'}| \left| \int \phi(f')\, d\overline{\mathrm{vol}} - \int f'\, d\mu \right| \\
&\leq \mathcal{M}|\mathcal{Q}_{E',n'}|\delta' \leq \delta/2.
\end{aligned}$$

And therefore using L^2 boundedness of ϕ, we thus have

$$\begin{aligned}
\left| \int \phi(f)\, d\overline{\mathrm{vol}} - \int f\, d\mu \right| &\leq \left\| \phi(f) - \phi(\tilde{f}) \right\|_2 + \left| \int \phi(\tilde{f})\, d\overline{\mathrm{vol}} - \int \tilde{f} d\mu \right| + \left\| \tilde{f} - f \right\|_2 \\
&\leq (C+1) \left\| f - \tilde{f} \right\|_2 + \frac{\delta}{2} \leq \delta.
\end{aligned} \tag{3.2}$$

For every good enough sofic approximation, from (3.1) and (3.2) we conclude

$$\mathrm{Hom}_C(\mathcal{Q},\Gamma,U',E',n',\delta',L) \subset \mathrm{Hom}_C(\mathcal{P},\Gamma,U,E,n,\delta,L).$$

Part 2:

Let $N \in \mathbb{N}$ be large enough such that $2^{-N-1} \leq \varepsilon/3$. Since $\mathcal{P}$ is generating, there exists a finite subset $D \subset \Gamma$ and a number $d \in \mathbb{N}$ such that corresponding to every $q \in \{ q_1, \ldots, q_N \}$, there is a

$$\tilde{q} = \sum_{f \in \mathcal{P}_{D,d}} c_{q,f} f$$

with $\|q - \tilde{q}\| \leq \varepsilon/6CN$. Set $\mathcal{M}_1 = \max_{q\in\{ q_1,\ldots,q_N \}} \max_{f\in\mathcal{P}_{D,d}} |c_{q,f}|$.

Let $\varepsilon' = \varepsilon/(3\mathcal{M}_1 N\, |\mathcal{P}_{D,d}|\, d2^{d+1})$. We show that if a finite set $E \subset \Gamma$ contains D, $n > d$ and $\delta \leq \varepsilon'$, then any two elements $\phi, \psi \in \mathrm{Hom}_C(\mathcal{Q},\Gamma,U',E',n',\delta',L) \subset \mathrm{Hom}_C(\mathcal{P},\Gamma,U,E,n,\delta,L)$ which are ε-separated with respect to $\rho_{\mathcal{Q}}$ are ε'-separated with respect to $\rho_{\mathcal{P}}$. In other words we show if ϕ and ψ are ε'-close, that is $\rho_{\mathcal{P}}(\phi,\psi) \leq \varepsilon'$, then $\rho_{\mathcal{Q}}(\phi,\psi) \leq \varepsilon$.

Observe that for any function of the form $f = \prod_{j=1}^d \alpha_{\gamma_j}(p_j) \in \mathcal{P}_{D,d}$, we have

$$\begin{aligned}
\|\phi(f) - \psi(f)\|_2 &= \left\| \prod_{j=1}^d \phi\circ\alpha_{\gamma_j}(p_j) - \prod_{j=1}^d \psi\circ\alpha_{\gamma_j}(p_j) \right\|_2 \leq \sum_{j=1}^d \left\|\phi\circ\alpha_{\gamma_j}(p_j) - \psi\circ\alpha_{\gamma_j}(p_j)\right\|_2 \\
&\leq \sum_{j=1}^d \left\|\phi\circ\alpha_{\gamma_j}(p_j) - L_{\gamma_j}\circ\phi(p_j)\right\|_2 + \left\|L_{\gamma_j}\big(\phi(p_j) - \psi(p_j)\big)\right\|_2 + \left\|L_{\gamma_j}\circ\psi(p_j) - \psi\circ\alpha_{\gamma_j}(p_j)\right\|_2.
\end{aligned}$$

As $\phi, \psi \in \mathrm{Hom}_C(\mathcal{P},\Gamma,U,E,n,\delta,L)$, the first and the last term in the above inequality are bounded by δ. Also, since we assumed $\rho_{\mathcal{P}}(\phi,\psi) \leq \varepsilon'$, if we have a good enough sofic approximation, the middle term is bounded above by $2^d\varepsilon'$ and we thus have

$$\|\phi(f) - \psi(f)\|_2 \leq d(2\delta + 2^d\varepsilon') \leq d2^{d+1}\varepsilon'.$$

Therefore one has

$$\begin{aligned}\|\phi(q_i) - \psi(q_i)\|_2 &\leq \|\phi(q_i - \tilde{q}_i)\|_2 + \|\psi(q_i - \tilde{q}_i)\|_2 + \|\phi(\tilde{q}_i) - \psi(\tilde{q}_i)\|_2 \\ &\leq 2C\,\|q_i - \tilde{q}_i\|_2 + \sum_{f\in\mathcal{P}_{D,d}} |c_{q_i,f}|\,\|\phi(f) - \psi(f)\|_2 \\ &\leq \varepsilon/3N + \mathcal{M}_1|\mathcal{P}_{D,d}|d2^{d+1}\varepsilon',\end{aligned}$$

whenever $1 \leq i \leq N$. Hence, we see that

$$\begin{aligned}\rho_{\mathcal{Q}}(\phi,\psi) = \sum_{i=1}^{\infty}\frac{1}{2^i}\,\|\phi(q_i) - \psi(q_i)\|_2 &\leq \Big(\sum_{i=1}^{N}\|\phi(q_i) - \psi(q_i)\|_2\Big) + \frac{1}{2^N} \\ &\leq N\Big(\frac{\varepsilon}{3N} + \mathcal{M}_1|\mathcal{P}_{D,d}|d2^{d+1}\varepsilon'\Big) + \varepsilon/3 \leq \varepsilon.\end{aligned}$$

Thus, from the two parts we conclude

$$N_{\varepsilon}\Big(\operatorname{Hom}_C(\mathcal{Q},\Gamma,U',E',n',\delta',L_i),\rho_{\mathcal{Q}}\Big) \leq N_{\varepsilon'}\Big(\operatorname{Hom}_C(\mathcal{P},\Gamma,U,E,n,\delta,L_i),\rho_{\mathcal{P}}\Big),$$

and consequently $h_{\operatorname{Hom}_C}(\mathcal{Q},\Gamma) \leq h_{\operatorname{Hom}_C}(\mathcal{P},\Gamma)$ and symmetrically the reverse inequality holds.

□

3.3 Entropy via Unital Positive Maps

Our definition of entropy used unital homomorphisms from $L^\infty(X,\mu)$ into $L^\infty(M,\overline{\mathrm{vol}})$. In this section we present an alternative approach by weakening the previous notion of "good models" to unital positive maps from $L^\infty(X,\mu) \to L^\infty(M,\overline{\mathrm{vol}})$. Continuing from the previous section, let $\alpha : G \times (X,\mu) \to (X,\mu)$ be a measure preserving action on a standard probability space (X,μ). Also, let $\mathcal{P} = \{\, p_i \,\}_{i=1}^\infty$ be a dynamically generating sequence in the unit ball of $L^\infty(X,\mu)$ and $\Gamma \subset G$ be a countable set generating for $\mathcal{P}$.

Definition 7. Let $n \in \mathbb{N}$, $\delta > 0$, $E \subset \Gamma$ be a finite subset and $U \subset G$ a precompact neighborhood of the identity. Suppose (M, vol, L) is a (U,β)-sofic approximation to G and let $\overline{\mathrm{vol}}$ denote the probability measure obtained after normalizing vol. Given $C \geq 1$, define $\mathrm{UP}_C(\mathcal{P},\Gamma,U,E,n,\delta,L)$ to be the set of all unital positive linear maps $\phi : L^\infty(X,\mu) \to L^\infty(M,\overline{\mathrm{vol}})$ satisfying:

1. $\|\phi(f)\|_2 \leq C\,\|f\|_2$ for all $f \in L^\infty(X,\mu)$,
2. $\phi\big(\prod_{i=1}^j \alpha_{\gamma_i}(p_i)\big) = \prod_{i=1}^j \phi\big(\alpha_{\gamma_i}(p_i)\big)$ for every $\prod_{i=1}^j \alpha_{\gamma_i}(p_i) \in \mathcal{P}_{E,n}$,
3. $\left|\int \phi(f)d\,\overline{\mathrm{vol}} - \int f d\mu\right| \leq \delta$ for all $f \in \mathcal{P}_{E,n}$,
4. $\|\phi \circ \alpha_g(f) - L_g \circ \phi(f)\|_2 \leq \delta$ for all $g \in U$ and $f \in \{\, p_1, p_2, \dots, p_n \,\}$.

Definition 8. Suppose $\Sigma = \{\, (M_i, \mathrm{vol}_i, L_i) \,\}_{i=1}^\infty$ is a sofic approximation sequence to G. Let $\mathcal{P}$ be a generating sequence in the unit ball of $L^\infty(X,\mu)$, $E \subset \Gamma$ a finite set,

$U \subset G$ a precompact neighborhood of the identity, $n \in \mathbb{N}$ and $\delta, \varepsilon > 0$. We define

$$h^{\varepsilon}_{\mathrm{UP}_C}(\mathcal{P}, \Gamma, U, E, n, \delta) = \limsup_{i\to\infty} \frac{1}{\mathrm{vol}_i(M_i)} \log N_\varepsilon\big(\,\mathrm{UP}_C(\mathcal{P}, \Gamma, U, E, n, \delta, L_i), \rho_{\mathcal{P}}\big),$$

$$h^{\varepsilon}_{\mathrm{UP}_C}(\mathcal{P}, \Gamma) = \inf_U \inf_E \inf_{n\in\mathbb{N}} \inf_{\delta>0} h^{\varepsilon}_{\mathrm{UP}_C}(\mathcal{P}, \Gamma, U, E, n, \delta),$$

$$h_{\mathrm{UP}_C}(\mathcal{P}, \Gamma) = \sup_{\varepsilon>0} h^{\varepsilon}_{\mathrm{UP}_C}(\mathcal{P}, \Gamma).$$

where the infimum in E is over all finite subsets of Γ and the infimum in U is over all precompact neighborhoods of the identity.

We will show that this notion of entropy h_{UP_C} using unital maps and the one in the previous section h_{Hom_C} using unital homomorphisms coincide. We will further show that they are independent of the constant C. We will need the following lemma from [OW87](II.1. Corollary 2) concerning continuity of the action on L^2 space:

Lemma 3.3.1. *Let $\alpha : G \times (X, \mu) \to (X, \mu)$ be a measure preserving action, then for every $f \in L^2(X, \mu)$, as $g_n \to g$ we have $\|\alpha_{g_n}(f) - \alpha_g(f)\|_2 \to 0$.*

Proof. One quick way to see this is through Koopman representation. We have a unitary representation $\pi : G \to U(L^2(X, \mu))$ given by $\pi(g)(f) = \alpha_g(f)$ whenever $g \in G$ and $f \in L^2(X, \mu)$. The unitary group $U(L^2(X, \mu))$ with strong operator topology is a Polish group i.e. a complete, separable metrizable group. And since π is a measurable homomorphism from a locally compact group into a Polish group, it is automatically continuous (see, for example [Kec95]). □

Theorem 3.3.2. *Let $\alpha : G \times (X, \mu) \to (X, \mu)$ be a measure preserving action of a nondiscrete locally compact sofic group G. Let $C \geq 1$, then,*

$$h_{\mathrm{Hom}_1}(\mathcal{P}, \Gamma) = h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma) = h_{\mathrm{UP}_C}(\mathcal{P}, \Gamma).$$

Proof. For every precompact $U \subset G$, finite set $F \subset \Gamma$, $n \in \mathbb{N}$ and $\varepsilon, \delta > 0$ we have the inclusions

$$\mathrm{Hom}_1(\mathcal{P}, \Gamma, U, F, n, \delta, L) \subset \mathrm{Hom}_C(\mathcal{P}, \Gamma, U, F, n, \delta, L) \subset \mathrm{UP}_C(\mathcal{P}, \Gamma, U, F, n, \delta, L)$$

and being equipped with same pseudometric $\rho_{\mathcal{P}}$ we see that

$$\begin{aligned} &N_\varepsilon\big(\,\mathrm{Hom}_1(\mathcal{P}, \Gamma, U, F, n, \delta, L)), \rho_{\mathcal{P}}\big) \\ &\qquad \le N_\varepsilon\big(\,\mathrm{Hom}_C(\mathcal{P}, \Gamma, U, F, n, \delta, L)), \rho_{\mathcal{P}}\big) \le N_\varepsilon\big(\,\mathrm{UP}_C(\mathcal{P}, \Gamma, U, F, n, \delta, L)), \rho_{\mathcal{P}}\big) \end{aligned}$$

and thus

$$h_{\mathrm{Hom}_1}(\mathcal{P}, \Gamma) \le h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma) \le h_{\mathrm{UP}_C}(\mathcal{P}, \Gamma).$$

So, to get our result we need to prove $h_{\mathrm{UP}_C}(\mathcal{P}, \Gamma) \le h_{\mathrm{Hom}_1}(\mathcal{P}, \Gamma)$. Towards that, let us fix a precompact neighborhood of the identity U, a finite set $F \subset \Gamma$, and $\varepsilon, \delta > 0$. Also, fix a large enough number $n \in \mathbb{N}$ such that $2^{-(n+1)} \le \varepsilon/4$.

Let $\eta_1, \eta_2, \eta > 0$ be some real numbers which we will specify later. Using Lemma 3.3.1 choose a neighborhood $W \subset G$ of the identity such that

$$\|\alpha_w(p) - p\|_2 \le \eta_1,$$

for every $p \in \{\, p_1, p_2, \ldots p_n \,\}$ and every $w \in W$. Since the action is measure preserving, we thus have

$$\|\alpha_{gw}(p) - \alpha_g(p)\|_2 \le \eta_1,$$

for all $g \in G$. By covering the precompact U with neighborhoods of the type $\{\, gW \,\}_{g \in U}$ and choosing some finite sub cover $\{\, gW \,\}_{g \in D}$ where $D \subset U$ is a finite subset we conclude that for every $g \in U$ there is a $d_g \in D$ with

$$\left\|\alpha_g(p) - \alpha_{d_g}(p)\right\|_2 \leq \eta_1 \tag{3.3}$$

for all $p \in \{ p_1, p_2, \dots p_n \}$.

Now, choose a finite partition of unity $\mathcal{Q} \subset L^\infty(X, \mu)$ consisting of projections such that every $f \in \mathcal{P}_{F,n} \cup \{ \alpha_d(p_i) : d \in D, 1 \leq i \leq n \}$ can be approximated in L^∞ norm with

$$\|f - \mathbb{E}(f \mid \mathcal{Q})\|_\infty \leq \eta_2$$

where $\mathbb{E}(\cdot \mid \mathcal{Q})$ denotes the conditional expectation operator. Since $\mathcal{P}$ is generating, by Kaplansky's density theorem there exists a finite set $E \subset \Gamma$ with $1_G \in E$ and a number $l \geq n$ such that every $q \in \mathcal{Q}$ has an approximation

$$\tilde{q} = \sum_{f \in \mathcal{P}_{E,l}} c_{q,f} f \in \operatorname{span}\{ \mathcal{P}_{E,l} \}$$

satisfying

$$\|q - \tilde{q}\|_2 \leq \eta^4.$$

Set $\mathcal{M} = \max_{q \in \mathcal{Q}} \max_{f \in \mathcal{P}_{E,l}} |c_{q,f}|$. There is a hierarchy of dependence between η_1, η_2 and η which one should be careful about in the remaining proof.

Having set up the approximations that we will need, coming to the crux, let $E' \subset \Gamma$ be a finite set containing E^2, a precompact set $U' \subset G$ be such that $UD \subset U'$; a number $n' \geq 2l$; and $\delta' > 0$ to be decided later. We will define a map

$$T : \mathrm{UP}_C(\mathcal{P}, \Gamma, U', E', n', \delta', L) \to \mathrm{Hom}_1(\mathcal{P}, \Gamma, U, F, n, \delta, L),$$

with the property that

$$\rho_{\mathcal{P}}(\phi,\psi)\geq\varepsilon \text{ implies } \rho_{\mathcal{P}}(T(\phi),T(\psi))\geq\varepsilon/2.$$

Suppose we have an element $\phi\in \mathrm{UP}_C(\mathcal{P},\Gamma,U',E',n',\delta',L)$. To define the map T, we first collect a few observations about ϕ :

i) If $q_1,q_2\in\mathcal{Q}$ are elements in the finite partition of unity $\mathcal{Q}$ and $\tilde{q}_1=\sum_{f\in\mathcal{P}_{E,l}} c_f\, f$, $\tilde{q}_2=\sum_{f\in\mathcal{P}_{E,l}}\tilde{c}_f\, f$ are their approximations, then since ϕ is multiplicative for the elements in $\mathcal{P}_{E,l}$ we have

$$\phi(\tilde{q}_1\tilde{q}_2)=\phi\Big(\sum_{f,g\in\mathcal{P}_{E,l}} c_f\tilde{c}_g\, fg\Big)=\sum_{f,g\in\mathcal{P}_{E,l}} c_f\tilde{c}_g\,\phi(f)\phi(g)=\phi(\tilde{q}_1)\phi(\tilde{q}_2).$$

ii) If $q\in\mathcal{Q}$, as $\mathcal{Q}$ consists of projections, we have $q^2=q$. Using the observation above, we see that the approximation $\tilde{q}\in\operatorname{span}\{\, P_{E,l}\,\}$ to q satisfies the bound,

$$\begin{aligned}\big\|\phi(\tilde{q})^2-\phi(\tilde{q})\big\|_2=\big\|\phi(\tilde{q}^2)-\phi(\tilde{q})\big\|_2&\leq\big\|\phi(\tilde{q}^2)-\phi(q^2)\big\|_2+\big\|\phi(q^2)-\phi(\tilde{q})\big\|_2\\&\leq 2C\eta^4+C\eta^4=3C\eta^4.\end{aligned}$$

Although $\phi(\tilde{q})\in L^\infty(M,\overline{\mathrm{vol}})$ will not in general be a projection, we will be able to say that on a set of large measure it takes values close to 0 or 1. Define $\mathscr{B}_q\subset M$ by

$$\mathscr{B}_q=\{\,\eta\leq\phi(\tilde{q})\leq 1-\eta\,\}\,,$$

we thus have

$$\eta^4\,\overline{\mathrm{vol}}(\mathscr{B}_q)\leq\int_{\mathscr{B}_q}\phi(\tilde{q})^2\big(1-\phi(\tilde{q})\big)^2\,d\overline{\mathrm{vol}}\leq\big\|\phi(\tilde{q})^2-\phi(\tilde{q})\big\|_2^2\,.$$

Therefore we conclude that for every $q \in \mathcal{Q}$,

$$\overline{\mathrm{vol}}(\mathscr{B}_q) \le 9C^2\eta^4. \tag{3.4}$$

iii) For every $q_1 \neq q_2 \in \mathcal{Q}$, as $\mathcal{Q}$ is a partition of unity consisting of projections, we have $q_1 q_2 = 0$. And therefore we have the bound

$$\int_M \phi(\tilde{q}_1)\phi(\tilde{q}_2)\, d\overline{\mathrm{vol}} \le \|\phi(\tilde{q}_1\tilde{q}_2)\|_2 = \|\phi(q_1q_2) - \phi(\tilde{q}_1\tilde{q}_2)\|_2 \le 2C\eta^4. \tag{3.5}$$

The observations above allow us to see the set $\{\, \phi(\tilde{q}) \,\}_{q\in\mathcal{Q}} \subset L^\infty(M, \overline{\mathrm{vol}})$ as an approximate partition of unity – using this we will piecewise build a homomorphism out of ϕ.

Given $q_j \in \mathcal{Q}$, define a set $A_j \subset M$ where $\phi(\tilde{q}_j)$ takes values close to 1 by

$$A_j = \{\, \phi(\tilde{q}_j) \ge 1 - \eta \,\}\,.$$

Since $\phi \in \mathrm{UP}_C(\mathcal{P}, \Gamma, U', E', n', \delta', L)$, it is δ'-measure preserving for the elements of $\mathcal{P}_{E,l}$ and thus

$$\left| \int \tilde{q}_j \, d\mu - \int \phi(\tilde{q}_j)\, d\overline{\mathrm{vol}} \right| \le \mathcal{M}\, |\mathcal{P}_{E,l}|\, \delta'.$$

Using the bound (3.4), we see that

$$
\begin{aligned}
\left|\mu(q_j) - \overline{\mathrm{vol}}(A_j)\right| &\le \|q_j - \tilde{q}_j\|_2 + \left|\int \tilde{q}_j \, d\mu - \int \phi(\tilde{q}_j)\, d\overline{\mathrm{vol}}\right| + \left|\int \phi(\tilde{q}_j)\ d\overline{\mathrm{vol}} - \int \chi_{A_j}\, d\overline{\mathrm{vol}}\right| \\
&\le \eta^4 + \mathcal{M}\,|\mathcal{P}_{E,l}|\,\delta' + \left|\int_{\phi(\tilde{q}_j)\le\eta} \phi(\tilde{q}_j)\, d\overline{\mathrm{vol}} + \int_{\mathscr{B}_{q_j}} \phi(\tilde{q}_j)\, d\overline{\mathrm{vol}}\right| \\
&\qquad\qquad\qquad\qquad + \left|\int_{A_j} \big(\phi(\tilde{q}_j) - 1\big)\, d\overline{\mathrm{vol}}\right| \\
&\le \eta^4 + \mathcal{M}\,|\mathcal{P}_{E,l}|\,\delta' + \eta + 9C^2\eta^4 + \eta \\
&= \mathcal{M}\,|\mathcal{P}_{E,l}|\,\delta' + 2\eta + (9C^2+1)\eta^4. \qquad (3.6)
\end{aligned}
$$

Also from the inequality (3.5), one has

$$
\overline{\mathrm{vol}}(A_i \cap A_j) = \int_M \chi_{A_i}\chi_{A_j}\, d\overline{\mathrm{vol}} \le \frac{1}{(1-\eta)^2}\int_M \phi(\tilde{q}_i)\phi(\tilde{q}_j)\, d\overline{\mathrm{vol}} \le 4C\eta^4, \qquad (3.7)
$$

whenever $i \ne j$. Therefore we have obtained “almost” disjoint sets A_i with measure $\overline{\mathrm{vol}}(A_i) \sim \mu(q_i)$. Further, we make them disjoint by defining subsets $B_j \subset A_j$ by

$$
B_j = A_j \setminus \cup_{k\ne j} A_k.
$$

We can now define corresponding to every map $\phi \in \mathrm{UP}_C(\mathcal{P}, \Gamma, U', E', n', \delta', L)$ the map $\widehat{\phi} \in \mathrm{Hom}_1(\mathcal{P}, \Gamma, U, F, n, \delta, L)$. Note that since our group is nondiscrete, we can assume (after removing a set of small measure if necessary) that the sofic approximation (M, vol, L) to G is a standard measure space with no atoms.

Choose a measure-space isomorphism $\widetilde{\phi} : M \to X$ from $(M, \overline{\mathrm{vol}})$ to (X, μ) satisfying: (i) if $\mu(q_j) \le \overline{\mathrm{vol}}(B_j)$ then $\chi_{\widetilde{\phi}(B_j)} \ge q_j$ and (ii) if $\mu(q_j) > \overline{\mathrm{vol}}(B_j)$, then $\chi_{\widetilde{\phi}(B_j)} \le q_j$. Let $\widehat{\phi} : L^\infty(X, \mu) \to L^\infty(M, \overline{\mathrm{vol}})$ be the isomorphism induced by $\widetilde{\phi}$.

We claim $\widehat{\phi} \in \mathrm{Hom}_1(\mathcal{P}, \Gamma, U, F, n, \delta, L)$. Since it is induced from an isomorphism from the space $(M, \overline{\mathrm{vol}})$ to (X, μ), it is a homomorphism from $L^\infty(X, \mu) \to L^\infty(M, \overline{\mathrm{vol}})$. Moreover, its L^2 norm is bounded by 1 and the δ-measure preserving condition is also automatically satisfied.

To show the δ-equivariance of $\widehat{\phi}$, we first estimate its L^2-distance from ϕ:

$$\left\|\phi(q_j) - \widehat{\phi}(q_j)\right\|_2 \leq \|\phi(q_j) - \phi(\tilde{q}_j)\|_2 + \left\|\phi(\tilde{q}_j) - \chi_{A_j}\right\|_2 + \left\|\widehat{\phi}(q_j) - \chi_{A_j}\right\|_2$$

whenever $q_j \in \mathcal{Q}$. The first term above is bounded by $C\eta^4$, the second term appears in inequality (3.6) above and similarly has a bound of $(2\eta + 9C^2\eta^4)^{1/2}$. To bound the last term, using the inequalities (3.7) and (3.6) we see that

$$\begin{aligned}
\left\|\widehat{\phi}(q_j) - \chi_{A_j}\right\|_2^2 &\leq \left\|\widehat{\phi}(q_j) - \chi_{B_j}\right\|_2^2 + \left\|\chi_{A_j} - \chi_{B_j}\right\|_2^2 \leq \left|\overline{\mathrm{vol}}(B_j) - \mu(q_j)\right| + 4C\left|\mathcal{Q}\right|\eta^4 \\
&\leq \left|\overline{\mathrm{vol}}(A_j) - \mu(q_j)\right| + 8C\left|\mathcal{Q}\right|\eta^4 \\
&\leq \mathcal{M}\left|\mathcal{P}_{E,l}\right|\delta' + 2\eta + (9C^2+1)\eta^4 + 8C\left|\mathcal{Q}\right|\eta^4 = \mathcal{M}\left|\mathcal{P}_{E,l}\right|\delta' + \mathcal{O}(\eta),
\end{aligned}$$

where $\mathcal{O}(\eta)$ means that the term goes to zero as η goes to zero (independently of $U', E', n', \delta', L, U, F, n, \delta$). Therefore putting them together we have

$$\left\|\phi(q_j) - \widehat{\phi}(q_j)\right\|_2 \leq \left(\mathcal{M}\left|\mathcal{P}_{E,l}\right|\delta' + \mathcal{O}(\eta)\right)^{1/2} + \mathcal{O}(\eta). \tag{3.8}$$

To check equivariance, let $g \in U$, and $i \leq n$. Assuming $d_g \in D$ is as in (3.3) and (M, vol, L) is a good enough sofic approximation, split in the following way:

$$\begin{aligned}
\left\|\widehat{\phi} \circ \alpha_g(p_i) - L_g \circ \widehat{\phi}(p_i)\right\|_2 \leq{}& \left\|\widehat{\phi} \circ \alpha_g(p_i) - \widehat{\phi} \circ \alpha_{d_g}(p_i)\right\|_2 + \left\|\widehat{\phi} \circ \alpha_{d_g}(p_i) - \phi \circ \alpha_{d_g}(p_i)\right\|_2 \\
&+ \left\|\phi \circ \alpha_{d_g}(p_i) - \phi \circ \alpha_g(p_i)\right\|_2 + \|\phi \circ \alpha_g(p_i) - L_g \circ \phi(p_i)\|_2 \\
&+ \left\|L_g \circ \phi(p_i) - L_g \circ \widehat{\phi}(p_i)\right\|_2.
\end{aligned}$$

The first and third term are bounded by η_1 and $C\eta_1$ respectively, the fourth term is bounded by δ' and for the second term from (3.8) we have

$$\begin{aligned}\left\|\widehat{\phi}\circ\alpha_{d_g}(p_i)-\phi\circ\alpha_{d_g}(p_i)\right\|_2 &\le \left\|\widehat{\phi}\circ\mathbb{E}(\alpha_{d_g}(p_i)\mid\mathcal{Q})-\phi\circ\mathbb{E}(\alpha_{d_g}(p_i)\mid\mathcal{Q})\right\|_2+(1+C)\eta_2\\ &\le |\mathcal{Q}|\left(\left(\mathcal{M}\,|\mathcal{P}_{E,l}|\,\delta'+\mathcal{O}(\eta)\right)^{1/2}+\mathcal{O}(\eta)\right)+\mathcal{O}(\eta_2).\end{aligned}$$

Similarly we can bound the last term assuming (M,vol,L) is a good enough sofic approximation and therefore we conclude that

$$\begin{aligned}\left\|\widehat{\phi}\circ\alpha_g(p_i)-L_g\circ\widehat{\phi}(p_i)\right\|_2 \le \mathcal{O}(\eta_1)+|\mathcal{Q}|\left(\left(\mathcal{M}\,|\mathcal{P}_{E,l}|\,\delta'+\mathcal{O}(\eta)\right)^{1/2}+\mathcal{O}(\eta)\right)&\\ +\mathcal{O}(\eta_2)+\delta' \;\le\; \delta,&\end{aligned}$$

by choosing the η's and δ' small enough.

Hence we get a map $T:\mathrm{UP}_C(\mathcal{P},\Gamma,U',E',n',\delta',L)\to\mathrm{Hom}_1(\mathcal{P},\Gamma,U,F,n,\delta,L)$ defined by $T(\phi)=\widehat{\phi}$. For proving ε-separability, suppose $\rho_{\mathcal{P}}(\widehat{\phi}_1,\widehat{\phi}_2)\le\varepsilon/2$, then

$$\begin{aligned}\rho_{\mathcal{P}}(\phi_1,\phi_2)&=\sum_{i=1}^{\infty}\frac{1}{2^i}\|\phi_1(p_i)-\phi_2(p_i)\|_2\le\sum_{i=1}^{n}\frac{1}{2^i}\|\phi_1(p_i)-\phi_2(p_i)\|_2+\frac{\varepsilon}{4}\\ &\le\sum_{i=1}^{n}\frac{1}{2^i}\Big(\left\|\widehat{\phi}_1(p_i)-\widehat{\phi}_2(p_i)\right\|_2+\left\|\widehat{\phi}_1(p_i)-\phi_1(p_i)\right\|_2+\left\|\widehat{\phi}_2(p_i)-\phi_2(p_i)\right\|_2\Big)+\frac{\varepsilon}{4}\\ &\le\rho_{\mathcal{P}}(\widehat{\phi}_1,\widehat{\phi}_2)+2\Big((1+C)\eta_2+|\mathcal{Q}|\big(\mathcal{M}\,|\mathcal{P}_{E,l}|\,\delta'+\mathcal{O}(\eta)\big)^{1/2}\Big)+\frac{\varepsilon}{4}\\ &\le\frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon,\end{aligned}$$

choosing small enough η's and δ'. So, the image of any ε-separated set under the map T is at least $\varepsilon/2$-separated. Hence, we can conclude

$$N_\varepsilon\big(\mathrm{UP}_C(\mathcal{P},\Gamma,U',E',n',\delta',L),\rho_{\mathcal{P}}\big)\le N_{\varepsilon/2}\big(\mathrm{Hom}_1(\mathcal{P},\Gamma,U,F,n,\delta,L),\rho_{\mathcal{P}}\big),$$

and therefore $h_{\mathrm{UP}_C}(\mathcal{P},\Gamma) \le h_{\mathrm{Hom}_1}(\mathcal{P},\Gamma)$.

$\square$

3.4 Entropy via Pseudometrics

In this section we present an approach to the entropy using space maps. This will be especially convenient since we will be able to drop the L^2-boundedness condition required by an operator algebraic approach. Also, it will readily provide a connection to the topological entropy. However, this approach requires a continuous action of the group on a compact metric space – which in general is not a loss of generality as compact models always exist for locally compact second countable groups:

Theorem 3.4.1 (Varadarajan [Var63]). *Let $G \curvearrowright (X,\mu)$ be a measure preserving action of a locally compact second countable group G on a standard probability space. Then, there is a compact metric space Y with a fully supported Borel probability measure ν on Y such that there exist a G-equivariant measure space isomorphism $(X,\mu) \to (Y,\nu)$.*

So, throughout this section let $\alpha : G \times (X,\mu) \to (X,\mu)$ be a continuous action of a sofic group G on the compact metric space X with a G-invariant probability measure μ. Also, let $\rho : X \times X \to \mathbb{R}$ be a bounded continuous pseudometric. We say that ρ is **generating** if for every $x, y \in X$ there is a $g \in G$ with $\rho\big(\alpha_g(x), \alpha_g(y)\big) > 0$.

Given a probability space $(M, \overline{\mathrm{vol}})$ and measurable maps $\phi, \psi : M \to X$ we define the psuedometric

$$\rho_2(\phi,\psi) = \Big(\int_M \rho\big(\phi(p),\psi(p)\big)^2 \, d\,\overline{\mathrm{vol}}(p) \Big)^{1/2}.$$

Definition 9. Let $U \subset G$ be a precompact neighborhood of the identity, $F \subset C(X)$

be a finite set, and $\delta > 0$. Suppose (M, vol, L) is a (U, η)- sofic approximation to G. Define $\mathrm{Map}(\rho, U, F, \delta, L)$ to be the set of all measurable maps $\phi : M \to X$ such that

i) $\rho_2(\phi \circ L_g, \alpha_g \circ \phi) < \delta$ for all $g \in U$,

ii) $\left| \int f(\phi(p))\ d\overline{\mathrm{vol}}(p) - \int f\ d\mu \right| < \delta$ for all $f \in F$.

Furthermore, given $C \geq 1$ define $\mathrm{Map}_C(\rho, U, F, \delta, L)$ to be the set of maps $\phi : M \to X$ that satisfy properties i) and ii) above and the additional property that

$$\|f \circ \phi\|_2 \leq C\, \|f\|_2 \ \text{ for every } f \in L^\infty(X, \mu).$$

Definition 10. Let $\Sigma = \{\ (M_i, \mathrm{vol}_i, L_i)_{i=1}^\infty\ \}$ be a sofic approximation to G and ρ be a generating pseudometric. Also, let $F \subset C(X)$ be a finite set, $U \subset G$ be a precompact neighborhood of the identity and $\delta, \varepsilon > 0$. We define

$$\begin{aligned} h_\mu^\varepsilon(\rho) &= \inf_{U \subset G} \inf_{F \subset C(X)} \inf_{\delta > 0} \limsup_{i \to \infty} \frac{1}{\mathrm{vol}_i(M_i)} \log N_\epsilon\big(\mathrm{Map}(\rho, U, F, \delta, L_i), \rho_2\big), \\ h(\mu, \rho) &= \sup_{\varepsilon > 0} h_\mu^\varepsilon(\rho), \end{aligned}$$

and, similarly

$$h_C(\mu, \rho) = \sup_{\varepsilon > 0} \inf_{U \subset G} \inf_{F \subset C(X)} \inf_{\delta > 0} \limsup_{i \to \infty} \frac{1}{\mathrm{vol}_i(M_i)} \log N_\epsilon\big(\mathrm{Map}_C(\rho, U, F, \delta, L_i), \rho_2\big),$$

where the infimum in F is over all finite subsets of $C(X)$ and U varies over all precompact neighborhoods of the identity in G.

We say that two pseudometrics ρ and ρ' are **uniformly equivalent** if given an $\varepsilon > 0$ there exists an $\varepsilon' > 0$ such that for any $x, y \in X$ if $\rho'(x, y) \leq \varepsilon'$ then $\rho(x, y) \leq \varepsilon$ and vice versa. And using Chebychev' inequality, for any two such uniformly equivalent continuous pseudometrics one can see that $h(\mu,\rho) = h(\mu,\rho')$ and $h_C(\mu,\rho) = h_C(\mu,\rho')$.

We show next that the two notions of the entropy $h(\mu,\rho)$ and $h_C(\mu,\rho)$ defined above give the same value.

Theorem 3.4.2. *Let G be a nondiscrete sofic group and $\alpha : G \times (X,\mu) \to (X,\mu)$ be a continuous action on a compact metric space (X,d) with a G-invariant probability measure μ. Let ρ be a continuous generating pseudometric for the action α, then,*

$$h_C(\mu,\rho) = h(\mu,\rho).$$

Proof. Since $\mathrm{Map}_1(\rho,U,F,\delta,L) \subset \mathrm{Map}_C(\rho,U,F,\delta,L) \subset \mathrm{Map}(\rho,U,F,\delta,L)$, we thus have

$$N_\varepsilon\big(\mathrm{Map}_1(\rho,U,F,\delta,L) \le N_\varepsilon\big(\mathrm{Map}_C(\rho,U,F,\delta,L),\rho_2\big) \le N_\varepsilon\big(\mathrm{Map}(\rho,U,F,\delta,L),\rho_2\big),$$

and hence $h_1(\mu,\rho) \le h_C(\mu,\rho) \le h(\mu,\rho)$.

So, we are left to prove $h(\mu,\rho) \le h_1(\mu,\rho)$. Without loss of generality, we may assume the diameter of ρ is at most 1. Fix a precompact neighborhood of the identity $U \subset G$, a finite set $F \subset C(X)$, and $\delta,\varepsilon > 0$. Set $\eta = \min\{\,\delta/8, \varepsilon/12\,\}$. Using the continuity of the action and compactness of X choose $0 < r < \eta$ such that for every $x,y \in X$ and every $g \in U$ if $d(x,y) \le r$ then,

$$\rho(\alpha_g(x),\alpha_g(y)) \le \eta;$$

and moreover $|f(x) - f(y)| \le \eta$ whenever $f \in F$.

Let $\{\,B_r(x_i)\,\}_{i=1}^N$ be a finite cover of X by open balls $B_r(x_i)$ of radius r. Set

$\zeta = \eta/4N^4$. For $1 \le i \le N$, choose open neighborhoods $U_i, V_i \subset X$ with $\bar{U}_i \subset V_i \subset \bar{V}_i \subset B_r(x_i)$ and $\mu(B_r(x_i)) - \mu(U_i) \le \zeta$ such that if $\{ f_i \}_{i=1}^N$ are continuous functions with

$$f_i(x) = \begin{cases} 1 & \text{if } x \in \bar{U}_i \\ 0 & \text{if } x \notin V_i \end{cases}$$

and

$$g_i(x) = \begin{cases} 1 & \text{if } x \in \bar{V}_i \\ 0 & \text{if } x \notin B_r(x_i) \end{cases},$$

then $\left| \int f_i - \int g_i \right| \le \zeta$. Fix such functions f_i, g_i.

Let $\mathcal{F}_N$ be the set of functions of the form $\prod_{i \in J} f_i$ where $J \subset \{ 1, 2, \dots N \}$. Similarly define $\mathcal{G}_N$ to be the set of products of the functions g_i for $i \in J \subset \{ 1, 2, \dots N \}$. If (M, vol, L) is a good enough sofic approximation and $\delta' = \min \{ \delta/14N^3, \varepsilon/18N^3 \}$ we will define a map

$$T : \mathrm{Map}(\rho, U, \mathcal{F}_N \cup \mathcal{G}_N, \delta', L) \to \mathrm{Map}_1(\rho, U, F, \delta, L).$$

So, let $\phi \in \mathrm{Map}(\rho, U, \mathcal{F}_N \cup \mathcal{G}_N, \delta', L)$. For every nonempty subset $J \subset \{ 1, 2, \dots N \}$, define

$$B_J = \cap_{i \in J} B_r(x_i),\ U_J = \cap_{i \in J} U_i \text{ and } V_J = \cap_{i \in J} V_i.$$

Our first step is to estimate $\overline{\mathrm{vol}}\big(\phi^{-1}(V_J)\big)$. Since f_i is zero outside V_i and 1 on U_i, we have

$$\int \prod_{i \in J} f_i \circ \phi = \overline{\mathrm{vol}}\big(\phi^{-1}(U_J)\big) + \int_{\phi^{-1}(V_J \setminus U_J)} \prod_{i \in J} f_i \circ \phi,$$

and using the property of ϕ we thus have

$$\left|\overline{\mathrm{vol}}\big(\phi^{-1}(U_J)\big) + \int_{\phi^{-1}(V_J\setminus U_J)} \prod_{i\in J} f_i\circ\phi - \int\prod_{i\in J} f_i\right| = \left|\int\prod_{i\in J} f_i\circ\phi - \int\prod_{i\in J} f_i\right| \le \delta'.$$

Similarly one obtains

$$\left|\overline{\mathrm{vol}}\big(\phi^{-1}(V_J)\big) + \int_{\phi^{-1}(B_J\setminus V_J)} \prod_{i\in J} g_i\circ\phi - \int\prod_{i\in J} g_i\right| = \left|\int\prod_{i\in J} g_i\circ\phi - \int\prod_{i\in J} g_i\right| \le \delta'.$$

Observe that we have the bounds

$$\begin{aligned}\overline{\mathrm{vol}}\big(\phi^{-1}(V_J)\big) \ge \overline{\mathrm{vol}}\big(\phi^{-1}(U_J)\big) + \int_{\phi^{-1}(V_J\setminus U_J)} \prod_{i\in J} f_i\circ\phi \\ \ge \Big(\int\prod_{i\in J} f_i\Big) - \delta' \ge \Big(\int\prod_{i\in J} g_i\Big) - |J|\zeta - \delta',\end{aligned}$$

and

$$\overline{\mathrm{vol}}\big(\phi^{-1}(V_J)\big) \le \int\prod_{i\in J} g_i - \int_{\phi^{-1}(B_J\setminus V_J)} \prod_{i\in J} g_i\circ\phi + \delta'.$$

We thus see from the above two inequalities that

$$\int_{\phi^{-1}(B_J\setminus V_J)} \prod_{i\in J} g_i\circ\phi \ \le |J|\zeta + 2\delta'.$$

Therefore we conclude that

$$\begin{aligned}\big|\overline{\mathrm{vol}}\big(\phi^{-1}(V_J)\big) - \mu(V_J)\big| &\le \left|\overline{\mathrm{vol}}\big(\phi^{-1}(V_J)\big) - \int\prod_{i\in J} g_i\right| + \left|\int\prod_{i\in J} g_i - \mu(V_J)\right| \\ &\le 2|J|\zeta + 3\delta'.\end{aligned}$$

Let $\{ A_i = V_i \setminus \cup_{j<i} V_j \}_{i=1}^N$ be the disjointification of V_i. Therefore, for every $i \leq N$, using the above inequality we have

$$\left|\overline{\mathrm{vol}}\big(\phi^{-1}(A_i)\big) - \mu(A_i)\right| \leq N^2(2N\zeta + 3\delta'). \tag{3.9}$$

We can now define our map $T : \mathrm{Map}(\rho, U, \mathcal{F}_N \cup \mathcal{G}_N, \delta', L) \to \mathrm{Map}_1(\rho, U, F, \delta, L)$. Note that since our group is nondiscrete, we can assume (after removing a set of small measure if necessary) that the sofic approximation (M, vol, L) to G is a standard measure space with no atoms.

Corresponding to $\phi \in \mathrm{Map}(\rho, U, \mathcal{F}_N \cup \mathcal{G}_N, \delta', L)$, let $T(\phi) = \widehat{\phi}$ where $\widehat{\phi} : M \to X$ is defined piecewise as follows: for $1 \leq i \leq N$, if $\mu(A_i) \geq \overline{\mathrm{vol}}\big(\phi^{-1}(A_i)\big)$ define $\widehat{\phi}$ on $\phi^{-1}(A_i)$ to be a measure space isomorphism onto some subset of A_i; else if $\mu(A_i) \leq \overline{\mathrm{vol}}(\phi^{-1}A_i)$, then choose some subset $C_i \subset \phi^{-1}(A_i)$ with $\overline{\mathrm{vol}}(C_i) = \mu(A_i)$ and define $\widehat{\phi}$ on C_i to be a measure space isomorphism onto A_i. Extend it to be a measure space isomorphism on the rest of M.

Observe that

$$\bigcup_{i=1}^N \mu(A_i) = \bigcup_{i=1}^N \mu(V_i) \geq \Big(\bigcup_{i=1}^N \mu\big(B_r(x_i)\big)\Big) - N\zeta = 1 - N\zeta.$$

Since $A_i \subset B_r(x_i)$, from (3.9) and the definition of $\widehat{\phi}$ it follows that there exists a subset $M' \subset M$ with $\overline{\mathrm{vol}}(M') \geq 1 - \big(N^3(2N\zeta + 3\delta') + N\zeta\big)$ such that for every $p \in M'$ the distance

$$d\big(\phi(p), \widehat{\phi}(p)\big) \leq r.$$

We show that $\widehat{\phi} \in \mathrm{Map}_1(\rho, U, F, \delta, L)$. We only need to check δ-equivariance as the other two properties are automatically satisfied since $\widehat{\phi}$ is measure preserving. So, let $g \in U$. If (M, vol, L) is a good enough approximation, by our choice of r we see that

$$\begin{aligned}
\rho_2(\widehat{\phi} \circ L_g, \alpha_g \circ \widehat{\phi}) &\leq \rho_2(\widehat{\phi} \circ L_g, \phi \circ L_g) + \rho_2(\phi \circ L_g, \alpha_g \circ \phi) + \rho_2(\alpha_g \circ \phi, \alpha_g \circ \widehat{\phi}) \\
&\leq \big(\eta + N^3(2N\zeta + 3\delta') + N\zeta\big) + \delta' + \big(\eta + N^3(2N\zeta + 3\delta') + N\zeta\big) \\
&\leq \delta.
\end{aligned}$$

Also observe that,

$$\rho_2(\phi, \widehat{\phi}) \leq \eta + \big(N^3(2N\zeta + 3\delta') + N\zeta\big) \leq \varepsilon/3.$$

Hence if $\rho_2(\widehat{\phi}, \widehat{\psi}) \leq \varepsilon/3$, then

$$\rho_2(\phi, \psi) \leq \rho_2(\widehat{\phi}, \phi) + \rho_2(\widehat{\phi}, \widehat{\psi}) + \rho_2(\widehat{\psi}, \psi) \leq \varepsilon.$$

Therefore T maps any ε-separated subset of $\mathrm{Map}(\rho, U, \mathcal{F}_N \cup \mathcal{G}_N, \delta', L)$ to an $\varepsilon/3$ separated subset of $\mathrm{Map}_1(\rho, U, F, \delta, L)$ and thus we conclude that

$$N_\varepsilon\big(\mathrm{Map}(\rho, U, \mathcal{F}_N \cup \mathcal{G}_N, \delta', L)\big) \leq N_{\varepsilon/3}\big(\mathrm{Map}_1(\rho, U, F, \delta, L)\big).$$

And hence $h(\mu, \rho) \leq h_1(\mu, \rho)$. $\square$

The proof of the next proposition essentially follows [KL13b][Prop 3.4].

Proposition 3.4.3. *Let $\alpha : G \times (X, \mu) \to (X, \mu)$ be a continuous action of a sofic group G on a compact metric space X with a G-invariant probability measure μ. Let ρ be a continuous generating pseudometric for the action α. Then*

$$h_C(\mu, \rho) = h_{\mathrm{Hom}_C}(\alpha).$$

Proof. Let $(X/\sim, \widetilde{\rho})$ be the quotient space modulo ρ. That is, $\sim$ is the equivalence relation such that for any $x, y \in X$ we say $x \sim y$ if and only if $\rho(x,y) = 0$; and $\widetilde{\rho}$ is the metric defined by $\widetilde{\rho}\big([x],[y]\big) = \rho(x,y)$ where $[x]$ denotes the equivalence class of x. Since ρ is continuous and X is a compact, the space $(X/\sim, \widetilde{\rho})$ is a compact metric space.

Let $\pi : X \to (X/\sim)$ be the canonical quotient map. Choose a sequence of continuous functions $\{\, q_i \,\}_{i=1}^{\infty} \in C\big(X/\sim, \widetilde{\rho}\big)$ with unit norm that separates points in $\big(X/\sim\big)$. Let $\mathcal{P} = \{\, p_i := q_i \circ \pi \,\}_{i=1}^{\infty} \subset C(X)$. Since ρ is a generating pseudometric we see that the family $\{\, p_i \circ \alpha_g : i \in \mathbb{N}, g \in G \,\}$ separates points in X and therefore by the Stone-Weierstrass Theorem, it generates $C(X)$ as a C^* algebra. Since $C(X)$ is L^2-dense in $L^\infty(X,\mu)$ the sequence $\mathcal{P} \subset L^\infty(X,\mu)$ is dynamically generating for the action α. Let $\Gamma \subset G$ be some countable set generating for $\mathcal{P}$ as in Lemma 3.2.1.

Let $\widetilde{\tau}$ be the metric on $(X/\sim)$ defined by

$$\widetilde{\tau}\big([x],[y]\big) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left| q_i\big([x]\big) - q_i\big([y]\big) \right| .$$

Since $X/\sim$ is compact, $\widetilde{\rho}$ and $\widetilde{\tau}$ are uniformly equivalent. If a continuous pseudometric τ on the space X is defined by $\tau(x,y) := \widetilde{\tau}\big([x],[y]\big)$ then we have that ρ and τ are uniformly equivalent. Therefore $h_C(\mu,\rho) = h_C(\mu,\tau)$. So it is enough to show $h_C(\mu,\tau) = h_{\mathrm{Hom}_C}(\alpha)$.

Given a measurable map $\phi : M \to X$ define the corresponding unital homomorphism

$$\widetilde{\phi} : L^\infty(X,\mu) \to L^\infty(M, \overline{\mathrm{vol}}),$$

by $\widetilde{\phi}(f) = f \circ \phi$ whenever $f \in L^\infty(X,\mu)$. Using Chebychev's inequality and the uniform continuity of the functions $\{ p_i \}_{i=1}^{\infty}$, observe that the pseudometrics

$$\rho_{\mathcal{P}}(\widetilde{\phi}, \widetilde{\psi}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left\| \widetilde{\phi}(p_i) - \widetilde{\psi}(p_i) \right\|_2 = \sum_{i=1}^{\infty} \frac{1}{2^i} \Big(\int \left| q_i\big([\phi(p)]\big) - q_i\big([\psi(p)]\big) \right|^2 d\overline{\mathrm{vol}}(p) \Big)^{1/2}$$

and

$$\tau_2(\phi, \psi) = \Big(\int \tau\big([\phi(p)], [\psi(p)]\big)^2 \, d\overline{\mathrm{vol}}(p) \Big)^{1/2}$$

are uniformly equivalent.

Now its clear that if δ' is small enough, for any $\phi \in \mathrm{Map}_C(\rho, U, \mathcal{P}_{E,n}, \delta', L)$ the corresponding homomorphism $\widetilde{\phi} \in \mathrm{Hom}_C(\mathcal{P}, \Gamma, U, E, n, \delta, L)$. And since τ_2 and $\rho_{\mathcal{P}}$ are uniformly equivalent, therefore $h_C(\mu, \tau) \leq h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma)$.

For the reverse inequality, note that from von Neumann's result(see [VW69]), it follows that for any L^2-bounded unital homomorphism $\Phi : L^\infty(X, \mu) \to L^\infty(M, \overline{\mathrm{vol}})$ there exists a corresponding space map $\phi : M \to X$ (neglecting measure zero sets) such that $\Phi = \widetilde{\phi}$. Using this correspondence observe that if $U \subset G$ a precompact neighborhood of the identity, $F \subset C(X)$ a finite set, and $\delta > 0$ are given then if $\Phi \in \mathrm{Hom}_C(\mathcal{P}, \Gamma, U, E, n, \delta', L)$ and δ' is small enough we see that

$$\tau_2(\alpha_g \circ \phi, \phi \circ L_g) \leq \delta$$

because τ_2 and $\rho_{\mathcal{P}}$ are uniformly equivalent. Moreover if $E \subset \Gamma$ and n are chosen such that every $f \in F$ can be approximated by $\tilde{f} = \sum_{g \in \mathcal{P}_{E,n}} c_f g$

$$\left\| f - \tilde{f} \right\|_\infty \leq \delta/3;$$

and we further see that

$$\left|\int f\circ\phi\, d\overline{\mathrm{vol}} - \int f\, d\mu\right| = \left|\int \Phi(f)\, d\overline{\mathrm{vol}} - \int f\, d\mu\right|$$
$$\leq \left|\int \Phi\big(f-\tilde{f}\big)\, d\overline{\mathrm{vol}}\right| + \left|\sum_{g\in\mathcal{P}_{E,n}} c_f\Big(\int \Phi(g)\, d\overline{\mathrm{vol}} - \int g\, d\mu\Big)\right| + \left|\int (\tilde{f}-f)\, d\mu\right|$$
$$\leq 2\delta/3 + |\mathcal{P}_{E,n}| \max\{\, c_f \,\}\, \delta' \leq \delta$$

if δ' is small enough. Therefore we have shown that to any $\Phi \in \mathrm{Hom}_C(\mathcal{P}, \Gamma, U, E, n, \delta', L)$ corresponds a $\phi \in \mathrm{Map}_C(\rho, U, F, \delta, L)$. And since τ_2 and $\rho_{\mathcal{P}}$ are uniformly equivalent we conclude that $h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma) \leq h_C(\mu, \tau)$.

□

Definition 11 (Measure Entropy)**.** Given the results of this chapter, we define the **measure entropy** $h^{meas}_{\Sigma}(\mu, \alpha)$ of a measure preserving action $\alpha : G\times(X, \mu) \to (X, \mu)$ with respect to a sofic approximation Σ to be equal to

$$h_{\mathrm{Hom}_C}(\mathcal{P}, \Gamma) = h_{UP_C}(\mathcal{P}, \Gamma)$$

for any generating sequence $\mathcal{P}$ in the unit ball of $L^\infty(X, \mu)$. Moreover if X is a compact metric space and ρ is any continuous generating pseudometric, then $h^{meas}_{\Sigma}(\mu, \alpha) = h(\mu, \rho)$.

Chapter 4

Topological Entropy

In this chapter we introduce topological entropy for continuous actions of locally compact sofic groups and prove the variational principle connecting measure entropy introduced in the previous chapter with topological entropy.

Throughout this chapter we will assume G to be a locally compact sofic group with a continuous action $\alpha : G \times X \to X$ on a compact metric space X and $\rho : X \times X \to \mathbb{R}$ to be a continuous bounded pseudometric. As before, we say that ρ is **generating** if for every $x, y \in X$ there is a $g \in G$ with $\rho\big(\alpha_g(x), \alpha_g(y)\big) > 0$.

Recall that given a probability space $(M, \overline{\mathrm{vol}})$ and measurable maps $\phi, \psi : M \to X$ we define the pseudometric

$$\rho_2(\phi, \psi) = \Big(\int_M \rho\big(\phi(p), \psi(p)\big)^2 \, d\,\overline{\mathrm{vol}}(p) \Big)^{1/2}.$$

4.1 Sofic Topological Entropy

Definition 12. Let $U \subset G$ be a precompact neighborhood of the identity and $\varepsilon, \delta > 0$. Suppose (M, vol, L) is a (U, η)- sofic approximation to G. Define $\mathrm{Map}(\rho, U, \delta, L)$ to be the set of all measurable maps $\phi : M \to X$ such that

$$\rho_2(\phi \circ L_g, \alpha_g \circ \phi) < \delta \text{ for all } g \in U.$$

For $\varepsilon > 0$, let $N_\epsilon(\mathrm{Map}(\rho, U, \delta, L), \rho_2)$ denote the maximum cardinality of an ϵ-separated subset of $\mathrm{Map}(\rho, U, \delta, L)$ with respect to ρ_2.

Definition 13. Let $\Sigma = \{ (M_i, \mathrm{vol}_i, L_i)_{i=1}^\infty \}$ be a sofic approximation to G and ρ be a generating pseudometric. We define

$$\begin{aligned} h^\varepsilon(\rho) &= \inf_{U \subset G} \inf_{\delta > 0} \limsup_{i \to \infty} \frac{1}{\mathrm{vol}_i(M_i)} \log N_\epsilon\big(\mathrm{Map}(\rho, U, F, \delta, L_i), \rho_2\big), \\ h(\rho) &= \sup_{\varepsilon > 0} h^\varepsilon(\rho) \end{aligned}$$

where infimum in U is over all precompact neighborhoods of the identity in G.

Theorem 4.1.1. *Let ρ and ρ' be bounded continuous generating pseudometrics on X. Then*

$$h(\rho) = h(\rho').$$

Proof. Without loss of generality we may assume that the diameters of ρ and ρ' are both bounded by 1. Let $F \in L^1(G, \lambda)$ be a positive function such that the $F > 0$ on all of G and $\|F\|_1 = 1$. Then define a metric d' on X by,

$$d'(x, y) = \int_G \rho'(\alpha_g(x), \alpha_g(y)) F(g) \, d\lambda(g),$$

and similarly define $d(x, y) = \int_G \rho(\alpha_g(x), \alpha_g(y)) F(g) \, d\lambda(g)$. It is clear that both d and d' are compatible metrics on X since ρ and ρ' are generating pseudometrics.

Fix $U \subset G$ a precompact neighborhood of the identity and positive reals $\delta, \varepsilon > 0$. We will show that if (M, vol, L) is a good enough sofic approximation then there exists a precompact neighborhood of the identity $U' \subset G$ and $\delta' > 0$ such that

$$\mathrm{Map}(\rho', U', \delta', L) \subset \mathrm{Map}(\rho, U, \delta, L).$$

Since ρ and ρ' are continuous, let $0 < \zeta \leq \varepsilon^2/2$ be such that for any $x, y \in X$ if $d'(x, y) \leq \zeta$ then $\rho(x, y) \leq \delta/2$; and similarly if $d(x, y) \leq \zeta$ then $\rho'(x, y) \leq \varepsilon/2$. Choose a compact neighborhood of identity $K \subset G$ with

$$\int_K F(g)\, d\lambda(g) \geq 1 - \frac{\zeta^2}{2}.$$

Let $U' \subset G$ be a precompact neighborhood of the identity with $KU \subset U'$ and set $\delta' = \min\left\{ \frac{\delta^2\zeta}{16}, \frac{\varepsilon^2\zeta}{8} \right\}$. Now, if $\phi \in \mathrm{Map}(\rho', U', \delta', L)$ and $g_0 \in U$, for a good enough sofic approximation we see that

$$\begin{aligned}
&\int\limits_M \int\limits_K \rho'\big(\alpha_g \circ \phi \circ L_{g_0}(p), \alpha_g \circ \alpha_{g_0} \circ \phi(p)\big)\, F(g)\, d\lambda(g)\, d\overline{\mathrm{vol}}(p) \\
&\qquad \leq \int\limits_M \int\limits_K \rho'\big(\alpha_g \circ \phi \circ L_{g_0}(p), \phi \circ L_g \circ L_{g_0}(p)\big)\, F(g)\, d\lambda(g)\, d\overline{\mathrm{vol}}(p) \\
&\qquad\qquad + \int\limits_M \int\limits_K \rho'\big(\phi \circ L_g \circ L_{g_0}(p), \alpha_g \circ \alpha_{g_0} \circ \phi(p)\big)\, F(g)\, d\lambda(g)\, d\overline{\mathrm{vol}}(p) \\
&\qquad \leq \int\limits_K (\delta' + \delta') F(g)\, d\lambda(g) \leq 2\delta'.
\end{aligned}$$

From Markov's inequality we thus have for the set

$$A := \left\{ p \in M \colon \int_K \rho'\big(\alpha_g \circ \phi \circ L_{g_0}(p), \alpha_g \circ \alpha_{g_0} \circ \phi(p)\big)\, F(g)\, d\lambda(g) \leq \frac{\zeta}{2} \right\},$$

that $\overline{\mathrm{vol}}(A) \geq 1 - 4\delta'/\zeta$. Moreover, observe from our choice of K, for every $p \in A$

$$d'\big(\phi \circ L_{g_0}(p), \alpha_{g_0} \circ \phi(p)\big) \leq \zeta,$$

and thus $\rho(\phi \circ L_{g_0}(p), \alpha_{g_0} \circ \phi(p)) \leq \delta/2$. Hence for every $g_0 \in U$ we conclude that

$$\begin{aligned}
\rho_2\big(\phi \circ L_{g_0}, \alpha_{g_0} \circ \phi\big) &= \Big(\int_M \rho\big(\phi \circ L_{g_0}(p), \alpha_{g_0} \circ \phi(p)\big)^2 \, d\overline{\mathrm{vol}}(p) \Big)^{1/2} \\
&\leq \frac{\delta}{2} + \left(\frac{4\delta'}{\zeta}\right)^{1/2} \leq \delta.
\end{aligned}$$

Therefore, $\mathrm{Map}(\rho', U', \delta', L) \subset \mathrm{Map}(\rho, U, \delta, L)$.

Set $\varepsilon' = \varepsilon^2\zeta/4$. Next we show any ε-separated set in $\mathrm{Map}(\rho', U', \delta', L)$ is ε'-separated in $\mathrm{Map}(\rho, U, \delta, L)$. So, suppose $\phi, \psi \in \mathrm{Map}(\rho', U', \delta', L)$ are such that $\rho_2(\phi, \psi) \leq \varepsilon'$ and observe that for every good enough sofic approximation (M, vol) we have

$$\begin{aligned}
d(\phi, \psi) &:= \int_M \int_G \rho\big(\alpha_g \circ \phi(p), \alpha_g \circ \psi(p)\big) F(g) \, d\lambda(g) \, d\overline{\mathrm{vol}}(p) \\
&\leq \int_M \int_K \rho(\alpha_g \circ \phi(p), \alpha_g \circ \psi(p)) F(g) \, d\lambda(g) \, d\overline{\mathrm{vol}}(p) \\
&\qquad\qquad + \int_M \int_{G \setminus K} \rho(\alpha_g \circ \phi(p), \alpha_g \circ \psi(p)) F(g) \, d\lambda(g) \, d\overline{\mathrm{vol}}(p) \\
&\leq \int_M \int_K \rho\big(\alpha_g \circ \phi(p), \phi \circ L_g(p)\big) F(g) \, d\lambda(g) \, d\overline{\mathrm{vol}}(p) \\
&\qquad + \int_M \int_K \rho\big(\phi \circ L_g(p), \psi \circ L_g(p)\big) F(g) \, d\lambda(g) \, d\overline{\mathrm{vol}}(p) \\
&\qquad\qquad + \int_M \int_K \rho\big(\psi \circ L_g(p), \alpha_g \circ \psi(p), \big) F(g) \, d\lambda(g) \, d\overline{\mathrm{vol}}(p) + \frac{\zeta^2}{2} \\
&\leq \delta' + \rho_2(\phi, \psi) + \delta' + \frac{\zeta^2}{2} \leq \varepsilon' + 2\delta' + \frac{\zeta^2}{2} \\
&\leq \frac{3\varepsilon^2\zeta}{4}.
\end{aligned}$$

And, Markov's inequality gives that if

$$B := \left\{ p \in M \colon d(\phi(p), \psi(p)) = \int_G \rho\big(\alpha_g \circ \phi(p), \alpha_g \circ \psi(p)\big) F(g)\, d\lambda(g)\, d\overline{\mathrm{vol}}(p) \leq \zeta \right\}$$

then $\overline{\mathrm{vol}}(B) \geq 1 - 3\varepsilon^2/4$; and thus for every point $p \in B$

$$\rho'\big(\phi(p), \psi(p)\big) \leq \varepsilon/2.$$

Thus we conclude that

$$\begin{aligned} \rho_2'\big(\phi, \psi\big) &= \left(\int_B \rho'(\phi(p), \psi(p))^2\, d\overline{\mathrm{vol}}(p) + \int_{M \setminus B} \rho'(\phi(p), \psi(p))^2\, d\overline{\mathrm{vol}}(p) \right)^{1/2} \\ &\leq \left(\frac{\varepsilon^2}{4} + \frac{3\varepsilon^2}{4} \right)^{1/2} = \varepsilon. \end{aligned}$$

Thus any subset of $\mathrm{Map}(\rho', U', \delta', L)$ which is ε - separated with respect to ρ_2' is ε' - separated with respect to ρ_2; and hence

$$\limsup_{i \to \infty} \frac{1}{\mathrm{vol}(M_i)} \log N_\varepsilon\big(\mathrm{Map}(\rho', U', \delta', L_i), \rho_2'\big) \leq \limsup_{i \to \infty} \frac{1}{\mathrm{vol}(M_i)} \log N_{\varepsilon'}\big(\mathrm{Map}(\rho, U, \delta, L_i), \rho_2\big).$$

Since we started with arbitrary U, ε, δ, this proves $h(\rho) \leq h(\rho')$ and symmetrically $h(\rho') \leq h(\rho)$ and therefore $h(\rho) = h(\rho')$. □

Definition 14 (Topological Entropy)**.** Using Theorem 4.1.1 we define the topological entropy $h_\Sigma^{top}(\alpha)$ of a continuous action $\alpha : G \times X \to X$ with respect to a sofic approximation Σ to be equal to $h(\rho)$ for any continuous generating pseudometric ρ.

4.2 Variational Principle

Theorem 4.2.1 (Variational Principle). *Suppose $\alpha : G \times X \to X$ is a continuous action of a locally compact sofic group G on a compact metric space X. For every sofic approximation Σ we have*

$$h_\Sigma^{top}(\alpha) = \sup_{\mu \in \mathscr{M}_G(X)} h_\Sigma^{meas}(\mu, \alpha),$$

where $\mathscr{M}_G(X)$ is the space of G-invariant Borel probability measures on X.

Proof. Let ρ be a compatible metric on X and $\mu \in \mathscr{M}_G(X)$. For every precompact neighborhood $U \subset G$, finite set $F \subset C(X)$ and $\delta, \varepsilon > 0$ we have the inclusion $\mathrm{Map}(\rho, U, F, \delta, L) \subset \mathrm{Map}(\rho, U, \delta, L)$. Thus

$$N_\varepsilon\big(\mathrm{Map}_\mu(\rho, U, F, \delta, L), \rho_2\big) \leq N_\varepsilon\big(\mathrm{Map}(\rho, U, \delta, L), \rho_2\big)$$

and hence $h(\mu, \rho) \leq h(\rho)$. Therefore, we conclude

$$\sup_{\mu \in \mathscr{M}_G(X)} h_\Sigma^{meas}(\mu, \alpha) \leq h_\Sigma^{top}(\alpha).$$

For the reverse inequality, let $e \in U_1 \subset U_2 \subset \cdots \subset U_n \subset \cdots$ be an increasing sequence of precompact neighborhoods of the identity in G and $1 \in F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$ be an increasing sequence of finite subsets of $C(X)$ such that their union is dense in $C(X)$. Using continuity of the action and compactness of X choose a finite subset $e \in D_n \subset U_n$ such that, given any $g \in U_n$ and $f \in F_n$ there is some $d_g \in D_n$ with

$$\big\|\alpha_g(f) - \alpha_{d_g}(f)\big\|_\infty \leq 1/n.$$

Moreover, let $0 < \delta_n < \big(1/n \sup_{f \in F_n} \|f\|_\infty\big)^2$ be chosen small enough that for every

$f \in F_n$ and any $x, y \in X$, $\rho(x,y) \le \delta_n$ implies $|f(x) - f(y)| \le 1/n$.

Because the space $\mathscr{M}(X)$ of all Borel probability measures on X is weak* compact, there exists a finite set $\mathcal{B}_n \subset \mathscr{M}(X)$ such that for every $\phi \in \mathrm{Map}(\rho, U_n, \delta_n, L)$ there is a measure $\mu_\phi \in \mathcal{B}_n$ with

$$\left| \int \alpha_d(f)\, d\mu_\phi - \int \alpha_d(f)\, d\phi_* \overline{\mathrm{vol}} \right| < 1/n, \tag{4.1}$$

for every $f \in F_n$ and $d \in D_n$.

Let $\mathcal{S}_n \subset \mathrm{Map}(\rho, U_n, \delta_n, L)$ be a maximal ε-separated set with respect to ρ_2. By the pigeonhole principle, there must be a measure $\mu_n \in \mathcal{B}_n$ such that

$$|\{\, \phi \in \mathcal{S}_n : \mu_\phi = \mu_n \,\}| \ge \frac{|\mathcal{S}_n|}{|\mathcal{B}_n|}.$$

Since every element of $\mathcal{S}_n$ is $\varepsilon-$separated, using (4.1) for $d = e \in G$, we thus have

$$N_\varepsilon\big(\mathrm{Map}_{\mu_n}(\rho, U_n, F_n, 1/n, L)\big) \ge \frac{|\mathcal{S}_n|}{|\mathcal{B}_n|} = \frac{1}{|\mathcal{B}_n|} N_\varepsilon\big(\mathrm{Map}(\rho, U_n, \delta_n, L)\big).$$

Therefore for every $n \in \mathbb{N}$, we conclude that

$$\begin{aligned} h^\varepsilon_{\mu_n}(\rho, U_n, F_n, 1/n) &= \limsup_{i\to\infty} \frac{1}{\mathrm{vol}(M_i)} \log N_\varepsilon\big(\mathrm{Map}_{\mu_n}(\rho, U_n, F_n, 1/n, L_i)\big) \\ &\ge \limsup_{i\to\infty} \frac{1}{\mathrm{vol}(M_i)} \log N_\varepsilon(\mathrm{Map}(\rho, U_n, \delta_n, L_i)) \\ &= h^\varepsilon(\rho, U_n, \delta_n). \end{aligned} \tag{4.2}$$

Observe that the sequence μ_n depends on ε and the idea now is to take the weak* limit point μ^ε of the sequence μ_n and show that

$$h(\mu^\varepsilon, \rho) \geq h^\varepsilon(\rho).$$

Furthermore, we check that μ_n's are invariant enough so that their limit point μ^ε is a G-invariant measure which will prove the variational principle.

Let $\phi \in \mathrm{Map}(\rho, U_n, \delta_n, L)$. For every $g \in U_n$, by Markov's inequality we have

$$\overline{\mathrm{vol}}\big(\{p \in M : \rho\big(\phi \circ L_g(p), \alpha_g \circ \phi(p)\big) \geq \delta_n^{1/2}\}\big) \leq \delta_n^{1/2}.$$

Thus, for every $f \in F_n$ by our choice of δ_n we see that

$$\left| \int f \circ \phi \circ L_g(p) - f \circ \alpha_g \circ \phi(p) \, d\overline{\mathrm{vol}}(p) \right| \leq \sup_{f \in F_n} \|f\|_\infty (\delta_n)^{1/2} + 1/n \leq 2/n.$$

And if (M, vol, L) is a good enough approximation, from the local measure isomorphism condition of the sofic approximation we have the bound

$$\left| \int f \circ \phi(p) - f \circ \phi\big(L_g(p)\big) \, d\overline{\mathrm{vol}}(p) \right| \leq 1/n.$$

Therefore, from the above two bounds we thus have

$$\left| \int f \circ \phi(p) - f \circ \alpha_g \circ \phi(p) \, d\overline{\mathrm{vol}}(p) \right| \leq 3/n.$$

Hence, for every $g \in U_n$, $f \in F_n$ and every μ_ϕ, using (4.1) we conclude that

$$
\begin{aligned}
&\left|\int f \circ \alpha_g(x) - f(x)\, d\mu_\phi(x)\right| \\
&\leq \left|\int f \circ \alpha_g(x) - f \circ \alpha_{d_g}(x)\, d\mu_\phi(x)\right| + \left|\int f \circ \alpha_{d_g}(x)\, d\mu_\phi(x) - \int f \circ \alpha_{d_g} \circ \phi(p)\, d\overline{\mathrm{vol}}(p)\right| \\
&\quad + \left|\int f \circ \alpha_{d_g} \circ \phi(p) - f \circ \phi(p)\, d\overline{\mathrm{vol}}(p)\right| + \left|\int f \circ \phi(p)\, d\overline{\mathrm{vol}}(p) - \int f(x)\, d\mu_\phi(x)\right| \\
&\qquad\qquad \leq 1/n + 1/n + 3/n + 1/n = 5/n. \qquad (4.3)
\end{aligned}
$$

Let $\mu^\varepsilon \in \mathscr{M}(X)$ be the weak* limit point of the sequence $\{\, \mu_n \,\}_{n=1}^\infty$. For any $f \in \cup_{n=1}^\infty F_n$ and $g \in G$, since

$$
\begin{aligned}
\left|\int f \circ \alpha_g - f\, d\mu^\varepsilon\right| \leq \left|\int f \circ \alpha_g\, d\mu^\varepsilon - \int f \circ \alpha_g\, d\mu_n\right| + \left|\int \big(f \circ \alpha_g - f\big)\, d\mu_n\right| \\
+ \left|\int f\, d\mu_n - \int f\, d\mu^\varepsilon\right|,
\end{aligned}
$$

using (4.3) and the definition of weak* limits, we thus have

$$
\int \big(f \circ \alpha_g - f\big)\, d\mu^\varepsilon = 0.
$$

Furthermore by our choice $\cup_{n=1}^\infty F_n$ is dense in $C(X)$, thus, μ^ε is G-invariant.

Now, using (4.2) we want to bound the measure entropy with respect to μ^ε from below with topological entropy. So, suppose a finite set $\mathcal{F} \subset C(X)$, a precompact neighborhood of the identity $\mathcal{U} \subset G$ and $\varepsilon, \delta > 0$ are given. Choose a large number n_0 with $\mathcal{U} \subset U_{n_0}$, $1/n_0 \leq \delta/4$ and such that every $f \in \mathcal{F}$ can be approximated by $\tilde{f} \in F_{n_0}$ with

$$
\left\|\tilde{f} - f\right\|_\infty \leq \delta/4,
$$

and moreover

$$\sup_{\tilde f\in F_{n_0}}\left|\int \tilde f\,d\mu_{n_0}-\int \tilde f\,d\mu^{\varepsilon}\right|\le \delta/4.$$

Observe that for every $\phi\in \mathrm{Map}_{\mu_{n_0}}(\rho,U_{n_0},F_{n_0},1/n_0,L)$, we have

$$\begin{aligned}
&\left|\int f\circ\phi\,d\overline{\mathrm{vol}}-\int f\,d\mu^{\varepsilon}\right|\\
&\qquad\le\left|\int\big(f\circ\phi-\tilde f\circ\phi\big)\,d\overline{\mathrm{vol}}\right|+\left|\int \tilde f\circ\phi\,d\overline{\mathrm{vol}}-\int\tilde f\,d\mu^{\varepsilon}\right|+\left|\int\tilde f\,d\mu^{\varepsilon}-\int f\,d\mu^{\varepsilon}\right|\\
&\qquad\le 2\frac{\delta}{4}+\left|\int\tilde f\circ\phi\,d\overline{\mathrm{vol}}-\tilde f\,d\mu_{n_0}\right|+\left|\int\tilde f\,d\mu_{n_0}-\tilde f\,d\mu^{\varepsilon}\right|\\
&\qquad\le 2(\delta/4)+1/n_0+\delta/4\le\delta.
\end{aligned}$$

Thus, we have the inclusion $\mathrm{Map}_{\mu_{n_0}}(\rho,U_{n_0},F_{n_0},1/n_0,L)\subset\mathrm{Map}_{\mu^{\varepsilon}}(\rho,\mathcal{U},\mathcal{F},\delta,L)$ and therefore using (4.2) we see that

$$h^{\varepsilon}_{\mu^{\varepsilon}}(\rho,\mathcal{U},\mathcal{F},\delta)\ge h^{\varepsilon}_{\mu_{n_0}}(\rho,U_{n_0},F_{n_0},1/n_0)\ge h^{\varepsilon}(\rho,U_{n_0},\delta_{n_0})\ge h^{\varepsilon}(\rho).$$

Since $\mathcal{U},\mathcal{F},\delta,\varepsilon$ were arbitrary, we conclude that

$$h(\mu^{\varepsilon},\rho)\ge h^{\varepsilon}(\mu^{\varepsilon},\rho)=\inf_{\mathcal{U}}\inf_{\mathcal{F}}\inf_{\delta>0}h^{\varepsilon}_{\mu^{\varepsilon}}(\rho,\mathcal{U},\mathcal{F},\delta)\ge h^{\varepsilon}(\rho),$$

and therefore

$$\sup_{\mu\in\mathscr{M}_G(X)}h^{meas}_{\Sigma}(\mu,\alpha)\ge\sup_{\varepsilon>0}h^{meas}_{\Sigma}(\mu^{\varepsilon},\alpha)=\sup_{\varepsilon>0}h(\mu^{\varepsilon},\rho)\ge\sup_{\varepsilon>0}h^{\varepsilon}(\rho)=h^{top}_{\Sigma}(\alpha).$$

$\square$

Chapter 5

Poisson Point Processes

In this chapter we show that the measure entropy of a Poisson point process on a nondiscrete locally compact sofic group is infinite. We start by defining Poisson point processes:

5.1 Notation and Definitions

Definition 15 (Poisson Point Process)**.** Let Y be a Polish space (completely metrizable separable space) with a Radon measure ν. A Poisson point process $(X, \mathbb{P}_\kappa)$ of intensity $\kappa > 0$ on (Y, ν) is defined as follows. The set X is the set of discrete subsets of Y with the σ-algebra on X is generated by sets of the form $\{ x \in X : |x \cap A| = k \}$, where $A \subset Y$ is measurable, of finite measure, and $k \in \mathbb{N} \cup \{ 0 \}$. Denoting by

$$N(A)(x) := |x \cap A|$$

the number of points of the discrete subset x in A, the probability measure $\mathbb{P}_\kappa$ on X is defined by

$$\mathbb{P}_\kappa(\{\, x \in X : N(A)(x) = k \,\}) = \frac{e^{-\kappa\nu(A)}\big(\kappa\nu(A)\big)^k}{k!}.$$

Further, given any disjoint collection of measurable sets $\{\, A_i \,\}_{i=1}^n \subset Y$, the events $\{\, x \in X : N(A_i)(x) = k_i \,\}$ are independent for any $k_i \in \mathbb{N} \cup \{\, 0 \,\}$ and

$$\begin{aligned}\mathbb{P}_\kappa\big(\{\, x \in X : N(A_i)(x) = k_i, 1 \leq i \leq n \,\}\big) &= \prod_{i=1}^n \mathbb{P}_\kappa(\{\, x \in X : N(A_i)(x) = k_i \,\}) \\ &= e^{(-\kappa\sum_{i=1}^n \nu(A_i))} \prod_{i=1}^n \frac{(\kappa\nu(A_i))^{k_i}}{k_i!}.\end{aligned}$$

We allow $n = \infty$. Using a variant of Kolmogorov's Extension Theorem, it can be shown that these conditions uniquely determine a Borel probability measure $\mathbb{P}$.

Throughout this chapter we assume G is a nondiscrete locally compact sofic group with a Haar measure λ. We will denote by $(X, \mathbb{P})$ a Poisson point process of intensity κ on (G, λ) – since we will be working with a single point process at a time, we will suppress κ in the notation and write just $\mathbb{P}$ instead of $\mathbb{P}_\kappa$. We will write χ_A^k for the indicator function of the set $\{\, x \in X : N(A)(x) = k \,\}$ and $\mathbb{P}(\chi_A^k) := \int_X \chi_A^k(x)\, d\mathbb{P}(x)$.

Given such a Poisson point process $(X, \mathbb{P})$ on the group (G, λ), there is a natural action $\alpha : G \times (X, \mathbb{P}) \to (X, \mathbb{P})$ where $g \in G$ acts by left translation on the discrete subset $x \in X$. The action α is measure preserving as

$$\alpha_g \{\, x \in X : N(A)(x) = k \,\} = \{\, x \in X : N(gA)(x) = k) \,\},$$

and since Haar measure is G-invariant we have $\mathbb{P}\big(\alpha_g \{\, x \in X : N(A)(x) = k \,\}\big) = \mathbb{P}\big(\{\, x \in X : N(A)(x) = k \,\}\big)$.

To avoid confusion, we write $\mathbb{N}_0 = \mathbb{N} \cup \{\, 0 \,\}$.

5.2 Entropy Calculations

Let $\{ B_i \}_{i=1}^{\infty} \subset G$ be a sequence of decreasing neighborhoods of the identity with $\cap_i B_i = \{e\}$. Since the sigma algebra of the Poisson process $(X, \mathbb{P})$ is generated by the functions χ_A^k where $k \in \mathbb{N}_0$ and $A \subset G$ is measurable, covering A with the translates of B_i's we can see that

$$\mathcal{Q} = \left\{ \chi_{B_i}^k : i \in \mathbb{N},\ k \in \mathbb{N}_0 \right\}, \tag{5.1}$$

is dynamically generating for the action α.

Fix $\{ q_i \}_{i=1}^{\infty} = \mathcal{Q}$ to be some ordering of the generating set $\mathcal{Q}$. Recall that given a finite set $E \subset G$ and $n \in \mathbb{N}$, the set $\mathcal{Q}_{E,n} \subset L^\infty(X, \mathbb{P})$ consists of functions of the form

$$\prod_{i \in J} \alpha_{g_i}(q_i)$$

where $J \subset \{1, \ldots, n\}$ and $g_1, \ldots, g_j \in E$. Define the set

$$\widetilde{\mathcal{Q}}_{E,n} = \mathcal{Q}_{E,n} \cup \left\{ \chi_{B_i}^1 \chi_{B_i^5}^1 : 1 \le i \le n \right\}.$$

Lemma 5.2.1. *There exist disjoint sets $\{ A_i \}_{i=1}^{I} \subset G$ and a number $K \in \mathbb{N}$ such that any $f \in \widetilde{\mathcal{Q}}_{E,n}$ is in the linear span of functions*

$$\prod_{i \in J} \chi_{A_i}^{k_i}$$

with coefficients 0 or 1, where J ranges over all subsets of $\{ 1, 2, 3, \ldots I \}$ and $0 \le k_i \le K$.

Proof. Suppose $f = \prod_{i \in J} \alpha_{g_i}(q_i)$. Then f is the characteristic function of

$$\{ x \in X : N(g_i B_i)(x) = k_i \ \forall i \in J \} .$$

Now if we look at the disjointification of $\{ g_i B_i \}_{i \in J}$ and write the above set using this disjointification we get that f can be written as a linear span of product of indicator functions of disjoint sets with coefficients 0 or 1. Similarly if $f = \chi^1_{B_i} \chi^1_{B_i^5}$, then we can write $f = \chi^1_{B_i} \chi^0_{B_i^5 \setminus B_i}$. Therefore, using this procedure for every $f \in \widetilde{\mathfrak{Q}}_{E,n}$ and taking their common refinement gives our result.

$\square$

For a finite subset $E \subset G$ and $n \in \mathbb{N}$, let $\mathcal{A}_{E,n}$ denote the span of the functions $\prod_{i \in J} \chi^{k_i}_{A_i}$ as given by the above lemma 5.2.1. Observe that $\widetilde{\mathfrak{Q}}_{E,n} \subset \mathcal{A}_{E,n}$.

We can now define the maps that will serve as "good models" to the action α. Suppose (M, vol, L) is a (V, η)-sofic approximation to G. Set

$$n_M = \lfloor \kappa \,\mathrm{vol}(M) \rfloor .$$

Let $(X_M, \mathbb{P}_M)$ be the space of subsets of M of cardinality n_M with the uniform measure $\mathbb{P}_M$. More precisely, define $\Phi : M^{n_M} \to X_M$ to be the map

$$\Phi(p_1, \dots, p_{n_M}) = \{ p_1, \dots, p_{n_M} \} ,$$

then $\mathbb{P}_M$ is the pushforward of the normalized product measure on M^{n_M} by Φ; that is

$$\mathbb{P}_M := \Phi_* \big(\overline{\mathrm{vol}} \times \cdots \times \overline{\mathrm{vol}} \big).$$

To be precise, Φ is only well-defined on the subset of M^{n_M} consisting of n_M-tuples of distinct elements. Because G is nondiscrete, after removing a set of small measure from M we may assume without loss of generality that the measure vol on M has no atoms. Therefore, the domain of Φ has full measure in M^{n_M} with respect to the product measure vol^{M_n}.

Given an $S \in X_M$, define the map $\phi_S : M \to X$ by

$$\phi_S(p) = \Big\{ \{g\}_{g \in V} \mid L(g,p) \in S \Big\}, \tag{5.2}$$

that is for every point p in the sofic approximation we look at its V neighborhood and collect the labels of the elements of S in V.

Now, let the map $\widetilde{\phi}_S : \mathcal{A}_{E,n} \to L^\infty(M, \overline{\mathrm{vol}})$ be defined such that if $f = \prod_{i \in J} \chi_{A_i}^{k_i}$ then $\widetilde{\phi}_S(f)$ is given by the indicator function of the set

$$\{\, p \in M : |\phi_S(p) \cap A_i| = k_i,\ \forall i \in J \,\}\,,$$

and, since $\mathcal{A}_{E,n}$ is the span of the functions $\prod_{i \in J} \chi_{A_i}^{k_i}$, extend $\widetilde{\phi}_S$ linearly to $\mathcal{A}_{E,n}$.

We will use these maps $\widetilde{\phi}_S$ to calculate the entropy of the Poisson point process – we first collect a few lemmas about them.

Lemma 5.2.2. *Let $\Sigma = \{\, (M_j, \mathrm{vol}_j) \,\}_{j \in \mathbb{N}}$ be a sequence of (V_j, η_j)-sofic approximations to G with V_j's increasing to G and $\eta_j \searrow 0$. Then for every $f \in \widetilde{\mathfrak{Q}}_{E,n}$ and $\delta > 0$ we have*

$$\lim_{j \to \infty} \mathbb{P}_{M_j}\Big(S \in X_{M_j} : \Big| \overline{\mathrm{vol}}_j\big(\widetilde{\phi}_S(f) \big) - \mathbb{P}(f) \Big| > \delta \Big) \to 0.$$

Proof. Using Lemma 5.2.1, any $f \in \widetilde{\mathfrak{Q}}_{E,n}$ can be written as the linear combination of the functions of the type $\prod_{i\in J}\chi_{A_i}^{k_i}$ and since

$$\begin{aligned}\mathbb{P}_{M_j}\Big(S\in X_{M_j}: \Big|\overline{\mathrm{vol}}\big(\,\widetilde{\phi}_S(f)\big)-\mathbb{P}(f)\Big|>\delta\Big)\\ \leq\sum_{J\subset I}\sum_{i\in J}\mathbb{P}_{M_j}\Big(S\in X_{M_j}: \Big|\overline{\mathrm{vol}}_j\big(\,\widetilde{\phi}_S(\chi_{A_i}^{k_i})\big)-\mathbb{P}(\chi_{A_i}^{k_i})\Big|>\frac{\delta}{|I|2^I}\Big),\end{aligned}$$

it is therefore sufficient to prove the lemma for functions of the form $\prod_{i\in J}\chi_{A_i}^{k_i}$. Recall that the A_i's are disjoint finite measure subsets of G.

So, choose a precompact neighborhood of the identity $U\subset G$ with $\cup_{i\in I}A_i\subset U$ and let j be large enough that $U^2\subset V_j$. Viewing $\overline{\mathrm{vol}}\Big(\,\widetilde{\phi}_S\big(\prod_{i\in J}\chi_{A_i}^{k_i}\,\big)\Big)$ as a random variable (because S is random with law $\mathbb{P}_{M_j}$), we estimate its expectation using Fubini's theorem:

$$\begin{aligned}\mathbb{E}\,\overline{\mathrm{vol}}_j\Big(\widetilde{\phi}_S\big(\prod_{i\in J}\chi_{A_i}^{k_i}\big)\Big)&=\int_{X_{M_j}}\overline{\mathrm{vol}}_j\,\{\,p\in M_j:|\,\phi_S(p)\cap A_i|=k_i,\forall i\in J\,\}\;d\mathbb{P}_{M_j}(S)\\ =\int_{M_j^o}&\mathbb{P}_{M_j}(S\in X_{M_j}:|\phi_S(p)\cap A_i|=k_i,\forall i\in J)\,d\overline{\mathrm{vol}}(p)\\ &+\int_{M_j\setminus M_j^o}\mathbb{P}_{M_j}(S\in X_{M_j}:|\phi_S(p)\cap A_i|=k_i,\forall i\in J)\,d\overline{\mathrm{vol}}(p)\\ \leq\Big(1-\sum_{i\in J}\rho_i\Big)^{(n_{M_j}-\sum_{i\in J}k_i)}&\frac{n_{M_j}!}{(n_{M_j}-\sum_{i\in J}k_i)!}\prod_{i\in J}\frac{\rho_i^{k_i}}{k_i!}+\eta_j\end{aligned}$$

where M_j^o denotes the good set of the sofic approximation (M_j,vol_j,L_j) (this means that for every $p\in M_j^o$, the map $u\mapsto L(u,p)$ is a measure-space isomorphism from V_j to a subset of M_j and the cocycle identity holds) and

$$\rho_i=\frac{\lambda(A_i)}{\mathrm{vol}(M_j)}.$$

The Poisson limit theorem gives us

$$\lim_{j\to\infty}\Big(1-\sum_{i\in J}\rho_i\Big)^{(n_{M_j}-\sum_{i\in J}k_i)}\frac{n_{M_j}!}{(n_{M_j}-\sum_{i\in J}k_i)!}\prod_{i\in J}\frac{\rho_i^{k_i}}{k_i!}=\prod_{i\in J}\mathbb{P}(\chi_{A_i}^{k_i}).$$

Since the A_i's are disjoint, we have

$$\lim_{j\to\infty}\mathbb{E}\,\overline{\mathrm{vol}}_j\Big(\widetilde{\phi}_S\Big(\prod_{i\in J}\chi_{A_i}^{k_i}\Big)\Big)=\mathbb{P}\Big(\prod_{i\in J}\chi_{A_i}^{k_i}\Big). \tag{5.3}$$

Similarly we calculate its second moment:

$$\begin{aligned}&\mathbb{E}\left(\overline{\mathrm{vol}}_j\Big(\widetilde{\phi}_S\big(\prod_{i\in J}\chi_{A_i}^{k_i}\big)\Big)\right)^2\\ &=\int_{M_j}\int_{M_j}\mathbb{P}_{M_j}\left(S\in X_{M_j}:|\phi_S(p_1)\cap A_i|=|\phi_S(p_2)\cap A_i|=k_i|,\forall i\in J\right)\,d\overline{\mathrm{vol}}(p_1)d\overline{\mathrm{vol}}(p_2),\end{aligned} \tag{5.4}$$

and observe if $p_1,p_2\in M_j^o$ and $p_2\notin L(U^2,p_1)$, then the integrand in (5.4) is equal to

$$\Theta_j=\Big(1-2\sum_{i\in J}\rho_i\Big)^{(n_{M_j}-2\sum_{i\in J}k_i)}\frac{n_{M_j}!}{(n_{M_j}-2\sum_{i\in J}k_i)!}\prod_{i\in J}(k_i!)^{-2}\rho_i^{2k_i}.$$

Thus, we see that

$$\begin{aligned}&\mathbb{E}\left(\overline{\mathrm{vol}}_j\Big(\widetilde{\phi}_S\big(\prod_{i\in J}\chi_{A_i}^{k_i}\big)\Big)\right)^2\\ &\quad\le(1-\eta_j)\left(1-\eta_j-\frac{\lambda(U^2)}{\mathrm{vol}(M_j)}\right)\Theta_j+(1-\eta_j)\left(\eta_j+\frac{\lambda(U^2)}{\mathrm{vol}(M_j)}\right)+\eta_j.\end{aligned}$$

By the Poisson limit theorem we have $\lim_{j\to\infty}\Theta_j=\mathbb{P}(\prod_{i\in J}\chi_{A_i}^{k_i})^2$. The second moment is always bounded from below by the square of the expectation and, hence

$$\lim_{j\to\infty} \mathbb{E}\left(\overline{\mathrm{vol}}_j\Big(\widetilde{\phi}_S\big(\prod_{i\in J}\chi_{A_i}^{k_i}\big)\Big)\right)^2 = \left(\mathbb{P}\Big(\prod_{i\in J}\chi_{A_i}^{k_i}\Big)\right)^2.$$

Therefore, using Chebyshev's inequality we conclude

$$\lim_{j\to\infty} \mathbb{P}_{M_j}\left(S\in X_{M_j} : \left|\overline{\mathrm{vol}}_j\Big(\widetilde{\phi}_S\big(\prod_{i\in J}\chi_{A_i}^{k_i}\big)\Big) - \mathbb{P}\Big(\prod_{i\in J}\chi_{A_i}^{k_i}\Big)\right| > \delta\right) \to 0.$$

$\square$

Lemma 5.2.3. *Let B be one of the decreasing neighborhoods $\{B_i\}_{i=1}^{\infty}\subset G$ of the identity and $k\in\mathbb{N}_0$. Let $n\in\mathbb{N}$ be large enough so that $\chi_B^k\in\mathcal{A}_{E,n}$. For every $\delta>0$ and $\zeta>0$ there exists a neighborhood $W\subset G$ of the identity such that if (M,vol,L) is a good enough sofic approximation, then for all $g\in W$ we have*

$$\mathbb{P}_M\big(S\in X_M : \big\|L_g\circ\widetilde{\phi}_S\circ\chi_B^k - \widetilde{\phi}_S\circ\chi_B^k\big\|_1 > \delta\big) < \zeta.$$

Proof. Let $D_1 = B\setminus gB$, $D_2 = B\cap gB$ and $D_3 = gB\setminus B$. Observe that

$$\mathbb{E}\left(\big\|L_g\circ\widetilde{\phi}_S\circ\chi_B^k - \widetilde{\phi}_S\circ\chi_B^k\big\|_1\right) = \int_{X_M}\overline{\mathrm{vol}}\big(L_g\circ\widetilde{\phi}_S(\chi_B^k)\,\Delta\,\widetilde{\phi}_S(\chi_B^k)\big)\,d\mathbb{P}_M(S),$$

and thus, if (M,vol,L) is a (V,η)-sofic approximation, using Fubini's theorem we see that

$$\begin{aligned}\mathbb{E}&\left(\big\|L_g\circ\widetilde{\phi}_S\circ\chi_B^k - \widetilde{\phi}_S\circ\chi_B^k\big\|_1\right)\\ &\le 2\int_{M^o}\mathbb{P}_M\big(S : |\phi_S(p)\cap B| = k, |\phi_S(p)\cap gB|\neq k\big)\,d\overline{\mathrm{vol}}(p) + \eta.\end{aligned}$$

Writing the integrand above using D_i's we have

$$\begin{aligned}
&\mathbb{E}\left(\left\|L_g \circ \widetilde{\phi}_S \circ \chi_B^k - \widetilde{\phi}_S \circ \chi_B^k\right\|_1\right) \\
&\leq 2\sum_{i=0}^{k} \int\limits_{M^o} \mathbb{P}_M\big(|\phi_S(p) \cap D_1| = i, |\phi_S(p) \cap D_2| = k-i, |\phi_S(p) \cap D_3| \neq i\big)\, d\overline{\mathrm{vol}}(p) + \eta \\
&= 2(1-\eta)\sum_{i=0}^{k} \binom{n_M}{i}\binom{n_M - i}{k-i}\rho_1^i \rho_2^{k-i}\Big(1 - \binom{n_M - k}{i}\rho_3^i\big(1 - \rho_3^{n_m - k - i}\big)\Big) + \eta,
\end{aligned}$$

where $\rho_i = \lambda(D_i)/\operatorname{vol}(M)$ for $1 \leq i \leq 3$.

Now, if g is chosen close to the identity, $\lambda(D_1)$ and $\lambda(D_3)$ can be made arbitrarily small and in turn ρ_i's too which leads to $\mathbb{E}\left(\left\|L_g \circ \widetilde{\phi}_S \circ \chi_B^k - \widetilde{\phi}_S \circ \chi_B^k\right\|_1\right)$ being arbitrarily small. Therefore, choosing $W \subset G$ appropriately, using Markov's inequality we conclude that if (M, vol, L) is good enough, then

$$\begin{aligned}
&\mathbb{P}_M\big(S \in X_M : \left\|L_g \circ \widetilde{\phi}_S \circ \chi_B^k - \widetilde{\phi}_S \circ \chi_B^k\right\|_1 > \delta\big) \\
&\qquad\qquad \leq \frac{1}{\delta}\,\mathbb{E}\left(\left\|L_g \circ \widetilde{\phi}_S \circ \chi_B^k - \widetilde{\phi}_S \circ \chi_B^k\right\|_1\right) \leq \zeta,
\end{aligned}$$

for all $g \in W$. □

Suppose $B \in \{\, B_i \,\}_{i=1}^{\infty} \subset G$ and n large enough so that $\chi_B^1 \chi_{B^5}^1 \in \mathcal{A}_{E,n}$. Then, for $\varepsilon > 0$, define

$$X_M(B, \varepsilon) := \Big\{ S \in X_M : \left\|\overline{\mathrm{vol}}\big(\widetilde{\phi}_S(\chi_B^1\, \chi_{B^5}^1)\big) - \mathbb{P}\big(\chi_B^1\, \chi_{B^5}^1\big)\right\|_1 \leq \varepsilon \Big\}.$$

Lemma 5.2.4. *Let $\varepsilon > 0$ and $B \in \{\, B_i \,\}_{i=1}^{\infty}$ be such that* $\exp(-\kappa\,\lambda(B^5)) \geq 1 - \varepsilon/3$. *Let $\varepsilon_0 = \varepsilon\kappa\lambda(B)/3$, then for every good enough sofic approximation (M, vol, L) and*

$S \in X_M(B, \varepsilon_0)$ we have

$$\log \mathbb{P}_M\big(T \in X_M : \left\|\widetilde{\phi}_S(\chi_B^1) - \widetilde{\phi}_T(\chi_B^1)\right\|_1 \leq \varepsilon_0\big)$$
$$\leq n_M(1-\varepsilon)\big(\log \kappa\lambda(B^2) - \log(1-\varepsilon)\big) + 4n_M H(\varepsilon) + \mathcal{O}\big(\log(n_M(1-\varepsilon))\big).$$

Proof. Let $Y_S \subset X_M$ be the set of $T \in X_M$ such that there exists a subset $S_0 \subset S$ with $|S_0| \geq (1-\varepsilon)\,|S|$ and an injective map $\beta : S_0 \to T$ with the property that for every $s \in S_0$,

$$\beta(s) \in L(B^2, s).$$

Then we have the following claim:

Claim 1. *For a good enough sofic approximation (M, vol, L) if $T \in X_M$ with*

$$\left\|\widetilde{\phi}_S(\chi_B^1) - \widetilde{\phi}_T(\chi_B^1)\right\|_1 \leq \varepsilon_0,$$

then $T \in Y_S$.

Proof. Let $S' \subset S$ be a subset consisting of the points $s \in S$ with $|L(B^4, s) \cap S| = 1$. Hence, S is B^2 separated, that is for every $s_1, s_2 \in S'$, $L(B^2, s_1)$ is disjoint from $L(B^2, s_2)$. Moreover, observe that

$$\widetilde{\phi}_{S'}(\chi_B^1) = \{p \in M : |L(B, p) \cap S'\}| = 1\} = \bigcup_{s \in S'} L(B, s)$$

and, for every point $p \in \widetilde{\phi}_S(\chi_B^1) \setminus \widetilde{\phi}_{S'}(\chi_B^1)$ we have $|L(B^5, p) \cap S| > 1$. If $\chi_{B^5}^{>1}$ denotes the indicator function of the set $\{x \in X : |x \cap B^5| > 1\}$, writing $\chi_B^1\, \chi_{B^5}^{>1} = \chi_B^1 - \chi_B^1 \chi_{B^5}^1$

we see that

$$\overline{\mathrm{vol}}\big(\widetilde{\phi}_S(\chi_B^1)\big) \leq \sum_{s\in S'} \overline{\mathrm{vol}}\big(L(B,s)\big) + \overline{\mathrm{vol}}\big(\widetilde{\phi}_S(\chi_B^1 - \chi_B^1\chi_{B^5}^1)\big)$$
$$= |S'|\,\frac{\lambda(B)}{\mathrm{vol}(M)} + \overline{\mathrm{vol}}\big(\widetilde{\phi}_S(\chi_B^1 - \chi_B^1\chi_{B^5}^1)\big). \tag{5.5}$$

Now, let $S_0 \subset S'$ be a subset such that for every $s_0 \in S_0$ the intersection

$$L(B,s_0) \cap \widetilde{\phi}_T(\chi_B^1) \neq \varnothing$$

is non empty. Then, we define the map $\beta : S_0 \to T$ by choosing

$$\beta(s_0) \in L(B,p)$$

for some $p \in L(B,s_0) \cap \widetilde{\phi}_T(\chi_B^1)$. It is clear that $\beta(s_0) \in L(B^2,s_0)$. And, assume for contradiction if $\beta(s_0) = \beta(s_0')$, then $s_0' \in L(B^4,s_0)$ but $s_0, s_0' \in S'$ and points in S' are B^2 separated. Hence, β is injective.

To show $|S_0| \geq (1-\varepsilon)\,|S|$, first observe that

$$\overline{\mathrm{vol}}\big(\,\widetilde{\phi}_S(\chi_B^1) \setminus \widetilde{\phi}_T(\chi_B^1)\big) \geq \overline{\mathrm{vol}}\big(\,\widetilde{\phi}_{S'}(\chi_B^1) \setminus \widetilde{\phi}_T(\chi_B^1)\big) = \overline{\mathrm{vol}}\Big(\bigcup_{s\in S'} L(B,s) \setminus \widetilde{\phi}_T(\chi_B^1)\Big)$$
$$\geq \overline{\mathrm{vol}}\Big(\bigcup_{s\in S'\setminus S_0} L(B,s) \setminus \widetilde{\phi}_T(\chi_B^1)\Big) = \lambda(B)\frac{|S' \setminus S_0|}{\mathrm{vol}(M)}.$$

Since $\left\|\widetilde{\phi}_S(\chi_B^1) - \widetilde{\phi}_T(\chi_B^1)\right\|_1 \leq \varepsilon_0$, using the above inequality and (5.5) we see that

$$|S_0| \geq |S'| - |S' \setminus S_0| \geq \frac{\mathrm{vol}(M)}{\lambda(B)}\Big(\overline{\mathrm{vol}}\big(\widetilde{\phi}_S(\chi_B^1\,\chi_{B^5}^1)\big) - \varepsilon_0\Big).$$

But, $S \in X_M(B, \varepsilon_0)$ thus we have

$$|S_0| \geq \frac{\mathrm{vol}(M)}{\lambda(B)}\Big(\mathbb{P}\big(\chi_B^1\, \chi_{B^5}^1\big) - 2\varepsilon_0\Big) = \frac{\mathrm{vol}(M)}{\lambda(B)}\Big(\mathbb{P}\big(\chi_B^1\big)\mathbb{P}\big(\chi_{B^5\setminus B}^0\big) - 2\varepsilon_0\Big),$$

and as $|S| = \kappa\lfloor \mathrm{vol}(M)\rfloor$, we conclude that

$$|S_0| \geq |S|\left(e^{-\kappa\lambda(B^5)} - 2\varepsilon/3\right) \geq (1-\varepsilon)\,|S|\,.$$

$\square$

In view of the claim, we estimate $\mathbb{P}_M(Y_S)$. Recall that the uniform measure $\mathbb{P}_M$ on X_M is the pushforward of vol under $\Phi : M^{n_M} \to X_M$. So, we consider $\Phi^{-1}(Y_S) \subset M^{n_M}$.

Given a subset $S_0 \subset S$ with cardinality $\lfloor(1-\epsilon)|S|\rfloor$ and an injective map $\sigma : S_0 \to \{\,1, \ldots, n_M\,\}$, let $Y(S_0, \sigma)$ to be the set of all $(p_1, \ldots, p_{n_M}) \in M^{n_M}$ such that

$$p_i \in L(B^2, s),$$

where $i = \sigma(s)$ for some $s \in S_0$. By our claim we have the inclusion

$$\Phi^{-1}(Y_S) \subset \bigcup_{S_0,\sigma} Y(S_0, \sigma).$$

Firstly, observe that if $\overline{\mathrm{vol}}^{n_M}$ denotes the product measure on M^{n_M} we see that

$$\overline{\mathrm{vol}}^{n_M}\big(Y(S_0,\sigma)\big) = \left(\frac{\lambda(B^2)}{\mathrm{vol}(M)}\right)^{|S_0|} \leq \left(\frac{\lambda(B^2)}{\mathrm{vol}(M)}\right)^{(1-\epsilon)n_M}.$$

Therefore, we have the estimate

$$\begin{aligned}\mathbb{P}_M(Y_S) = \overline{\mathrm{vol}}^{n_M}\big(\Phi^{-1}(Y_S)\big) &\le \sum_{S_0,\,\sigma} \overline{\mathrm{vol}}^{n_M}\big(Y(S_0,\sigma)\big)\\ &\le \Big(\frac{\lambda(B^2)}{\mathrm{vol}(M)}\Big)^{(1-\epsilon)n_M} \binom{n_m}{\lfloor(1-\varepsilon)n_M\rfloor} \frac{n_M!}{(n_M - \lfloor(1-\varepsilon)n_M\rfloor)!}.\end{aligned}$$

Writing $H(\varepsilon) = -\varepsilon\log\varepsilon - (1-\varepsilon)\log(1-\varepsilon)$ and using the entropy bound $\binom{n}{k} \le \exp\big(2n\,H(n/k)\big)$ for binomial coefficients we see that

$$\mathbb{P}_M(Y_S) \le \Big(\frac{\lambda(B^2)}{\mathrm{vol}(M)}\Big)^{(1-\epsilon)n_M} \exp(4n_M H(\varepsilon))\big(\lfloor(1-\varepsilon)n_M\rfloor\big)!.$$

Since $n_M = \lfloor\kappa\,\mathrm{vol}(M)\rfloor$, using Stirling's formula we conclude that

$$\begin{aligned}&\log\mathbb{P}_M(Y_S)\\ &\quad\le n_M(1-\varepsilon)\Big(\log\Big(\frac{\lambda(B^2)}{\mathrm{vol}(M)}\Big) + \log\big(n_M(1-\varepsilon)\big)\Big) + 4n_M H(\varepsilon) + \mathcal{O}(\log(n_M(1-\varepsilon)))\\ &\quad\le n_M(1-\varepsilon)\big(\log\kappa\lambda(B^2) - \log(1-\varepsilon)\big) + 4n_M H(\varepsilon) + \mathcal{O}\big(\log(n_M(1-\varepsilon))\big).\end{aligned}$$

□

Theorem 5.2.5. *Let G be a non-discrete, locally compact sofic group. The Poisson point process $(X,\mathbb{P})$ on G has infinite measure entropy.*

Proof. Let $\mathcal{Q}$ be a dynamically generating sequence as in (5.1) and $\Gamma \subset G$ a countable set generating for $\mathcal{Q}$ as in the Lemma 3.2.1. Let $n \in \mathbb{N}$, $\delta,\varepsilon > 0$, $\{e\} \in E \subset \Gamma$ be a finite set and $U \subset G$ be a precompact neighborhood of the identity. For $S \in X_M$, let $\widetilde{\phi}_S : \mathcal{A}_{E,n} \to L^\infty(M,\overline{\mathrm{vol}})$ be the map as defined in the beginning of the section. Fix some $\zeta > 0$.

Using Lemma 5.2.3 and Lemma 3.3.1 find a neighborhood $W \subset G$ along with a subset $X_M^1 \subset X_M$ of measure $\mathbb{P}_M(X_M^1) > 1 - \zeta$ such that for every $S \in X_M^1$ and $g \in W$ we have

$$\left\| L_g \circ \widetilde{\phi}_S(q_i) - \widetilde{\phi}_S(q_i) \right\|_2 \leq \frac{\delta}{4} \text{ and } \|\alpha_g(q_i) - q_i\|_2 \leq \frac{\delta}{4}, \tag{5.6}$$

whenever $1 \leq i \leq n$. Using compactness of the closure of U, extract a finite set $D \subset U$ such that, for every $g \in U$ there is a $d_g \in D$ with $g \in d_g W$.

Given $S \in X_M$, analogous to $\widetilde{\phi}_S$ define a map $\widetilde{\psi}_S : \mathcal{A}_{E \cup D, n} \to L^\infty(M, \overline{\mathrm{vol}})$ and extend it to all of $L^\infty(X, \mathbb{P})$ by defining

$$\widetilde{\psi}_S(f) = \widetilde{\psi}_S \circ \mathbb{E}(f \mid \mathcal{A}_{E \cup D, n}),$$

for every $f \in L^\infty(X, \mathbb{P})$ and where $\mathbb{E}(\cdot \mid \mathcal{A}_{E \cup D, n}\})$ is the conditional expectation operator.

Let $C = 2$. We show that for all good enough sofic approximation (M, vol, L) there is a subset $X_M^o \subset X_M$ of large measure such that

$$\widetilde{\psi}_S \in UP_C(\mathcal{Q}, \Gamma, U, E, n, \delta, L),$$

whenever $S \in X_M^o$.

1) Suppose $f \in \mathcal{Q}_{E,n}$, then by Lemma 5.2.2 there is subset $X_M^2 \subset X_M$ of measure $\mathbb{P}_M(X_M^2) > 1 - \zeta$ with

$$|\widetilde{\psi}_S(f) - \mathbb{P}(f)| \leq \delta,$$

whenever $S \in X_M^2$. Hence, $\widetilde{\psi}_S$ is δ-measure preserving. Moreover, by the proof of the Lemma 5.2.2, whenever $S \in X_M^2$, the map $\widetilde{\psi}_S$ is δ-measure preserving on $\mathcal{A}_{E,n}$

and, therefore $\left\|\widetilde{\psi}_S(f)\right\|_2 \leq (1+\delta)\|f\|_2$ for every $f \in \mathcal{A}_{E,n}$. Since the conditional expectation being a projection operator is bounded and thus we have

$$\left\|\widetilde{\psi}_S(f)\right\|_2 = \left\|\widetilde{\psi}_S \circ \mathbb{E}(f \mid \mathcal{A}_{E\cup D,n})\right\|_2 \leq (1+\delta)\|f\|_2 < C\|f\|_2,$$

for every $f \in L^\infty(X,\mathbb{P})$ and $S \in X_M^2$.

2) Observe that if $f = \prod_{i\in J}\chi_{A_i}^{k_i} \in \mathcal{A}_{E\cup D,n}$ and $g = \prod_{i\in J'}\chi_{A_i}^{k_i} \in \mathcal{A}_{E\cup D,n}$ then as $\{ A_i \}$ are disjoint we see that $\widetilde{\psi}_S(fg) = \widetilde{\psi}_S(f)\widetilde{\psi}(g)$. Since any $h \in \mathfrak{Q}_{E,n}$ can be written as a linear combination of functions of the form $f = \prod_{i\in J}\chi_{A_i}^{k_i}$ with $\{ 0,1 \}$ coefficients, thus we have

$$\widetilde{\psi}_S(h) = \widetilde{\psi}_S\big(\prod_{i=1}^{n}\alpha_{\gamma_i}(q_i)\big) = \prod_{i=1}^{n}\widetilde{\psi}_S(\alpha_{\gamma_i}(q_i)).$$

3) We will show that for all $g \in U$ and $q \in \{ q_1, q_2, \ldots q_n \}$,

$$\left\|\widetilde{\psi}_S \circ \alpha_g(q) - L_g \circ \widetilde{\psi}_S(q)\right\|_2 \leq \delta,$$

whenever $S \in X_M^1 \cap X_M^2$. Let (M, vol, L) be a (V,η)-sofic approximation with $U^2 \subset V$. Observe that for $q = \chi_B^k$, and $\gamma \in E\cup D$ we get the inclusion

$$\begin{aligned}\widetilde{\psi}_S \circ \alpha_\gamma(q) &\subset \{ p \in M_o : |\psi_S(p) \cap \alpha_\gamma(B)| = k \} \cup M \setminus M_o \\ &= \{ L(\gamma,p) : p \in M_o, |\psi_S(p) \cap B| = k \} \cup M \setminus M_o \\ &= L_\gamma \circ \widetilde{\psi}_S(q) \cup M \setminus M_o.\end{aligned}$$

Similarly one obtains $L_\gamma \circ \widetilde{\psi}_S(q) \subset \widetilde{\psi}_S \circ \alpha_\gamma(q) \cup M \setminus M_o$. Thus, we have

$$\left\| \widetilde{\psi}_S \circ \alpha_\gamma(q) - L_\gamma \circ \widetilde{\psi}_S(q) \right\|_2 \leq (2\eta)^{1/2}. \tag{5.7}$$

Now, for every $S \in X_M^1 \cap X_M^2$, $g \in U$ and $q \in \{ q_1, \ldots, q_n \}$ we have

$$\begin{aligned}\left\| \widetilde{\psi}_S \circ \alpha_g(q) - L_g \circ \widetilde{\psi}_S(q) \right\|_2 &\leq \left\| \widetilde{\psi}_S \circ \alpha_g(q) - \widetilde{\psi}_S \circ \alpha_{d_g}(q) \right\|_2 + \left\| \widetilde{\psi}_S \circ \alpha_{d_g}(q) - L_g \circ \widetilde{\psi}_S(q) \right\|_2 \\ &= \left\| \widetilde{\psi}_S\big(\alpha_g(q) - \alpha_{d_g}(q)\big) \right\|_2 + \left\| \widetilde{\psi}_S \circ \alpha_{d_g}(q) - L_g \circ \widetilde{\psi}_S(q) \right\|_2,\end{aligned}$$

and using (5.7) and (5.6) we see that

$$\begin{aligned}\left\| \widetilde{\psi}_S \circ \alpha_g(q) - L_g \circ \widetilde{\psi}_S(q) \right\|_2 &\leq C \left\| \alpha_g(q) - \alpha_{d_g}(q) \right\|_2 + \left\| L_{d_g} \circ \widetilde{\psi}_S(q) - L_g \circ \widetilde{\psi}_S(q) \right\|_2 + (2\eta)^{1/2} \\ &\leq \left\| L_{d_g} \circ \widetilde{\psi}_S(q) - L_g \circ \widetilde{\psi}_S(q) \right\|_2 + C\delta/4 + (2\eta)^{1/2}.\end{aligned}$$

Furthermore, using the η-measure preserving property of the action L_g on the space $(M, \overline{\mathrm{vol}})$, we have the bound

$$\begin{aligned}\left\| \widetilde{\psi}_S \circ \alpha_g(q) - L_g \circ \widetilde{\psi}_S(q) \right\|_2 &\leq \left\| L_{d_g^{-1}}\big(L_{d_g} \circ \widetilde{\psi}_S(q) - L_g \circ \widetilde{\psi}_S(q)\big) \right\|_2 + \frac{c\delta}{4} + 2(2\eta)^{1/2} \\ &\leq \left\| \widetilde{\psi}_S(q) - L_{d_g^{-1} g} \circ \widetilde{\psi}_S(q) \right\|_2 + C\delta/4 + 4(2\eta)^{1/2},\end{aligned}$$

and finally using (5.6) we conclude

$$\left\| \widetilde{\psi}_S \circ \alpha_g(q) - L_g \circ \widetilde{\psi}_S(q) \right\|_2 \leq (1 + C)\delta/4 + 5(2\eta)^{1/2} = 3\delta/4 + 4(2\eta)^{1/2}.$$

So, if η is small enough, we get the desired $\left\| \widetilde{\psi}_S \circ \alpha_g(q) - L_g \circ \widetilde{\psi}_S(q) \right\|_2 \leq \delta$.

Hence, for every $S \in X_M^1 \cap X_M^2$ and good enough sofic approximation, we have

$$\widetilde{\psi}_S \in UP_C(\mathcal{Q}, \Gamma, U, E, n, \delta, L).$$

Recall that $q_i \in \mathcal{Q}$ are defined by balls B_i with $\lambda(B_i) \searrow 0$. For $\varepsilon > 0$, let i_o be large enough that B_{i_o} satisfy the condition in the Lemma 5.2.4 and ε_0 be as provided by the same lemma. Now, choose $\tilde{\varepsilon} > 0$ small enough with the property that for any maps $\widetilde{\psi}_S, \widetilde{\psi}_T$ with

$$\rho_{\mathcal{Q}}(\widetilde{\psi}_S, \widetilde{\psi}_T) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left\| \widetilde{\psi}_S(q_i) - \widetilde{\psi}_S(q_i) \right\|_2 \leq \tilde{\varepsilon}$$

implies $\left\| \widetilde{\psi}_S(\chi^1_{B_{i_0}}) - \widetilde{\psi}_T(\chi^1_{B_{i_0}}) \right\|_1 \leq \varepsilon_0$.

Set $X_M^o = X_M^1 \cap X_M^2 \cap X_M(B, \varepsilon_0)$. The Lemma 5.2.2 ensures that $\mathbb{P}_M\big(X_M(B, \varepsilon_0)\big) > 1 - \zeta$ whenever (M, vol, L) is good enough sofic approximation. Therefore, $\mathbb{P}_M(X_M^o) > 1 - 3\zeta$.

From Lemma 5.2.4 and our choice of $\tilde{\varepsilon}$ we have a bound on the measure of radius $\tilde{\varepsilon}$-ball around $\widetilde{\psi}_S$ with respect to $\rho_{\mathcal{Q}}$; thus, covering $\{\widetilde{\psi}_S : S \in X_M^o\}$ with such balls we see that

$$\begin{aligned}
h^{\tilde{\varepsilon}}(\mathcal{Q}, \Gamma, U, E, n, \delta, L) &= \limsup_{j \to \infty} \frac{1}{\mathrm{vol}(M_j)} \log N_{\tilde{\varepsilon}}\big(UP_c(\mathcal{Q}, \Gamma, U, E, n, \delta, L)\big) \\
&\geq \limsup_{j \to \infty} \frac{-1}{\mathrm{vol}(M_j)} \Big(n_{M_j}(1 - \varepsilon)\big(\log \kappa \lambda(B_{i_0}^2) - \log(1 - \varepsilon)\big) + 4 n_{M_j} H(\varepsilon) + \mathcal{O}\big(\log(n_{M_j}(1 - \varepsilon))\big)\Big) \\
&\qquad + \limsup_{j \to \infty} \frac{\log(1 - 3\zeta)}{\mathrm{vol}(M_j)},
\end{aligned}$$

and using $n_{M_j} = \lfloor \kappa \, \mathrm{vol}(M_j) \rfloor$ we thus have

$$h^{\tilde{\varepsilon}}(\mathcal{Q}, \Gamma, U, E, n, \delta, L) \geq -\kappa(1 - \varepsilon)\big(\log \kappa \lambda(B_{i_0}^2) + \log(1 - \varepsilon)\big) - 4\kappa H(\varepsilon).$$

Since $\lambda(B^2_{i_o})$ can be arbitrarily small we conclude $h(\mathcal{Q},\Gamma) = \sup_{\tilde{\varepsilon}} h^{\tilde{\varepsilon}}(\mathcal{Q},\Gamma) = \infty$. $\square$

Chapter 6

Restricted Action of a Lattice

The purpose of this chapter is to establish the relation between the entropies of an action of a sofic group G and the restriction of the action to a lattice $\Gamma < G$.

In the case of the group of integers, where an action is specified by a measure space automorphism $T : (X, \mu) \to (X, \mu)$, the action of a subgroup $n\mathbb{Z} < \mathbb{Z}$ corresponds to the automorphism T^n and we have a well known relationship

$$h_\mu(T^n) = |n|\, h_\mu(T)$$

between the Kolmogorov-Sinai entropies. Our aim is to generalize this relationship to locally compact sofic groups.

6.1 Induced Sofic Approximation

First, we recall the notion of an induced sofic approximation from a lattice to the ambient group. This is the content of Prop 2.0.3 from Chapter 2 – however, we will rephrase it using fundamental domains.

Let $\Gamma < G$ be a lattice in a locally compact group G. Suppose $F \subset G$ is a right fundamental domain for G/Γ with the identity $e \in F$. Let λ be a normalized bi-invariant Haar measure on G with $\lambda(F) = 1$. Define the cocycle $\beta : G \times F \to \Gamma$ by $\beta(g, f) = \gamma \in \Gamma$ to be the unique element γ such that $gf\gamma \in F$. Observe that

$$\beta(gh, f) = \beta(h, f)\beta\big(g, hf\beta(g, f)\big).$$

Moreover, $\beta(f, e) = e$ for every $f \in F$ and $\beta(\gamma, e) = \gamma^{-1}$ for every $\gamma \in \Gamma$.

Suppose Γ is sofic – using the construction of an induced action from a lattice to the ambient group we define a sofic-approximation to G induced from a sofic-approximation to Γ. Let $U \subset G$ be a precompact neighborhood of the identity and $\varepsilon > 0$. Set $\varepsilon' = 1 - \sqrt{1 - \varepsilon^2}$ and let $F_0 \subset F$ be a compact set with $\lambda(F_0) > (1 - \varepsilon')\lambda(F) = 1 - \varepsilon'$. Since the cocycle β is continuous and U is precompact, the image of β restricted to $U \times F_0$ is a finite subset, which we denote by

$$E = \{\gamma \in \Gamma : \exists u \in U, \exists f \in F_0 \text{ such that } \beta(u, f) = \gamma\}.$$

Let $(M, \mathrm{vol}, \sigma)$ be an (E', ε') sofic approximation to Γ such that $E^{-1}E^{-1} \subset E'$. That is $(M, \mathrm{vol}, \sigma)$ is a finite graph with vertex set M equipped with the counting measure vol, and a subset $M_0 \subset M$ along with a map $\sigma : E' \times M \to M$ satisfying the properties in Def.4. Let $(F \times M, \lambda \times \mathrm{vol})$ be a finite measure space with the product measure. (We can extend σ to $\Gamma \times M$ arbitrarily).

We define a map $L : U \times (F \times M) \to (F \times M)$ which will give us a sofic approximation to G. Given a $g \in U$, $f \in F_0$ and $p \in M$, let

$$L(g, (f, p)) = \big(chr)\big),$$

and for $f \in F \setminus F_0$, let $L(g,(f,p)) = \big(gf\beta(g,f),p\big)$. (We can extend L to $G \times (F \times M)$ arbitrarily).

It is clear that $L(e,(f,p)) = (f,p)$. Also, using the cocycle condition of β observe that for every $f \in F_0$ and $p \in M_0$ we have

$$\begin{aligned} L(g, L(h,(f,p))) &= L\big(g, (hf\beta(h,f), \sigma(\beta(h,f)^{-1},p))\big) \\ &= (ghf\beta(h,f)\beta(g,hf\beta(h,f)), \sigma(\beta(g,hf\beta(h,f))^{-1}, (\sigma(\beta(h,f)^{-1},p)))) \\ &= (ghf\beta(gh,f), \sigma(\beta(gh,f)^{-1}),p) = L(gh,(f,p)). \end{aligned}$$

We next show that for every $f \in F_0$ and $p \in M_0$ the map $L(\,\cdot\,,(f,p)) : U \to F \times M$ is a measure space isomorphism onto its image. Suppose $V \subset U$ be a measurable subset. For any $\gamma \in E'$, let

$$V_\gamma = \{g \in V : \beta(g,f) = \gamma\}.$$

Observe that V is a disjoint union $V = \sqcup_{\gamma \in E'} V_\gamma$. If $\gamma \neq \gamma' \in E'$, since $p \in M_0$ then $\sigma(\gamma^{-1},p) \neq \sigma((\gamma')^{-1},p)$ and therefore

$$L(V_\gamma,(f,p)) = (gf\gamma, \sigma(\gamma^{-1},p)) \text{ and } L(V_{\gamma'},(f,p)) = (gf\gamma', \sigma((\gamma')^{-1},p))$$

are disjoint. This gives us that the measure of the image of V under the map $L(\cdot,(f,p))$ is

$$\lambda \times \mathrm{vol}\,\big(L(V,(f,p))\big) = \bigsqcup_{\lambda \in E'} \lambda \times \mathrm{vol}\,\big(L(V_\gamma,(f,p))\big) = \bigsqcup_{\lambda \in E'} \lambda(V_\gamma f \gamma) = \bigsqcup_{\lambda \in E'} \lambda(V_\gamma) = \lambda(V).$$

Hence $L(\,\cdot\,,(f,p)) : U \to F \times M$ is a measure space isomorphism onto its image. Therefore if (M,vol) is an (E',ε') sofic approximation to Γ then, $(F \times M, \lambda \times \mathrm{vol})$ is

(U, ε) sofic approximation to G.

So, given a sofic approximation Σ to Γ, we denote by $\widetilde{\Sigma}$ the induced sofic approximation to the ambient group G.

6.2 Connections between $h(\alpha_\Gamma)$ and $h(\alpha_G)$

Theorem 6.2.1. *Let $\Gamma < G$ be a lattice in a locally compact group G with a Haar measure λ. Let Σ be a sofic approximation to Γ and $\widetilde{\Sigma}$ be the induced sofic approximation to G. Suppose $\alpha : G \times (X, \mu) \to (X, \mu)$ is a measure preserving action and $\alpha_\Gamma : \Gamma \times (X, \mu) \to (X, \mu)$ its restriction. Then,*

$$h_\Sigma^{meas}(\mu, \alpha_\Gamma) \leq \lambda(G/\Gamma)\, h_{\widetilde{\Sigma}}^{meas}(\mu, \alpha),$$

where $\lambda(G/\Gamma)$ is the covolume of the lattice Γ.

Remark 4. Note that scaling the Haar measure λ by a factor C scales the entropy by the factor $1/C$, therefore, the term on the right is independent of the chosen λ.

Proof. By normalizing Haar measure we assume that $\lambda(G/\Gamma) = 1$. Using the existence of compact models 3.4.1 for an action, we can assume that (X, μ) is a compact metric space with an invariant measure μ and a compatible metric $\rho : X \times X \to [0, 1]$.

Let $U \subset G$ be a precompact neighborhood, $\mathscr{F} \subset C(X)$ be a finite set and $\delta, \varepsilon > 0$. Let $(M, \mathrm{vol}, \sigma)$ be a sofic approximation to Γ and $(F \times M, \lambda \times \mathrm{vol}, L)$ be the corresponding induced sofic approximation to G. We will show that there exist finite sets $E' \subset \Gamma$, $\mathscr{F}' \subset C(X)$ and positive reals $\delta', \varepsilon' > 0$ such that for every good enough sofic approximation we have,

$$N_\varepsilon(\mathrm{Map}(\rho, E', \mathscr{F}', \delta', \sigma)) \leq N_{\varepsilon'}(\mathrm{Map}(\rho, U, \mathscr{F}, \delta, L)).$$

Moreover ε' does not depend on $U, \mathscr{F}, \delta$.

Choose a compact set $F_0 \subset F$ with measure $\lambda(F_0) > (1-\delta^2/3)\lambda(F) = (1-\delta^2/3)$. Let $E \subset \Gamma$ be the finite image of the cocycle β restricted to $\bar{U} \times F_0$. Then using continuity of the action choose a positive real $\zeta > 0$ such that if for $x, y \in X$,

$$\rho(x,y) \leq \zeta \text{ then } \rho(\alpha_g(x), \alpha_g(y)) \leq \delta/3 \tag{6.1}$$

for **every** $g \in \bar{U} F_0 E$. Using compactness of F_0 and continuity of the action α obtain a finite subset $\{f_i\}_{i=1}^n \subset F_0$ such that for every $h \in \mathscr{F}$ and $f \in F_0$, there is some f_i, $1 \leq i \leq n$ with

$$\|h \circ \alpha_f - h \circ \alpha_{f_i}\|_\infty \leq \delta/6. \tag{6.2}$$

Set $E' = E^{-1}$, $\mathscr{F}' = \cup_{h\in\mathscr{F}} \cup_{i=1}^n h \circ \alpha_{f_i}$ and $\delta' = \min\{\zeta^2, \delta^2/3\,|E'|\}$. Assuming $\phi \in \mathrm{Map}(\rho, E', \mathscr{F}', \delta', \sigma)$ we have the bound

$$\rho_2(\phi\circ\sigma_\gamma, \alpha_\gamma\circ\phi)^2 = \int_M \rho^2(\phi(\sigma_\gamma(p), \alpha_\gamma(\phi(p)))\, d\overline{\mathrm{vol}}(p) \leq (\delta')^2$$

whenever $\gamma \in E'$. Apply Markov's inequality to get a subset

$$M_\gamma = \{p \in M : \rho(\phi(\sigma_\gamma(p), \alpha_\gamma(\phi(p))) \leq (\delta')^{1/2}\},$$

with $\overline{\mathrm{vol}}(M_\gamma) \geq (1-\delta')$. Set $M' = \cap_{\gamma\in E'} M_\gamma$.

Given $f \in F$ and $p \in M$, corresponding to ϕ define the map $\widetilde{\phi} : F \times M \to X$ by

$$\widetilde{\phi}(f,p) = \alpha_f(\phi(p)).$$

We claim $\widetilde{\phi} \in \mathrm{Map}(\rho, U, \mathcal{F}, \delta, L)$. Firstly, let us check the δ-equivariance condition.

For any group element $g \in U$ we see that

$$\begin{aligned}
&\rho_2(\widetilde{\phi} \circ L_g, \alpha_g \circ \widetilde{\phi})^2 \\
&\qquad = \int\limits_M \int\limits_F \rho^2(\widetilde{\phi}(L_g(f,p)), \alpha_g(\widetilde{\phi}(f,p)))\ d\lambda(f) d\overline{\mathrm{vol}}(p) \\
&\qquad \leq \int\limits_{F_0} \int\limits_{M'} \rho^2(\alpha_{gf\beta(g,f)}(\phi(\sigma_{\beta(g,f)^{-1}}(p))), \alpha_{gf}(\phi(p)))\ d\lambda(f) d\overline{\mathrm{vol}}(p) + \delta^2/3 + |E'|\,\delta' \\
&\qquad \leq (\delta/3)^2 + \delta^2/3 + \delta^2/3 \leq \delta^2,
\end{aligned}$$

using the uniform estimate (6.1) for elements of the type $gf\beta(g,f) \in \bar{U}F_0E$.

Secondly, we verify the δ - m.p. condition. Given a continuous function $h \in \mathscr{F}$, since the action α is measure preserving, applying (6.2) we see that

$$\begin{aligned}
&\left| \int_F \int_M h(\widetilde{\phi}(f,p)) d\overline{\mathrm{vol}}(p) d\lambda(f) - \int_X h(x) d\mu(x) \right| \\
&\quad = \left| \int_F \int_M h \circ \alpha_f(\phi(p))\ d\overline{\mathrm{vol}}(p) d\lambda(f) - \int_X h \circ \alpha_f(x) d\mu(x) \right| \\
&\qquad \leq \left| \int_{F_0} \int_M h \circ \alpha_{f_i}(\phi(p))\ d\overline{\mathrm{vol}}(p) d\lambda(f) - \int_X h \circ \alpha_{f_i}(x) d\mu(x) \right| + 2\delta/6 + \delta^2/3 \\
&\qquad \leq \delta' + \delta/3 + \delta^2/3 \leq \delta.
\end{aligned}$$

Therefore,

$$\mathrm{Map}(\rho, E', \mathscr{F}', \delta', \sigma)) \subset \mathrm{Map}(\rho, U, \mathscr{F}, \delta, L).$$

In order to finish the proof, we have to show that there exists a positive real $\varepsilon' > 0$ with the property that any ε-separated ϕ and ψ in $\mathrm{Map}(\rho, E', \mathscr{F}', \delta', \sigma))$ leads to ε'-separated $\widetilde{\phi}$ and $\widetilde{\psi}$ in $\mathrm{Map}(\rho, U, \mathscr{F}, \delta, L)$.

Let $B \subset F$ be a symmetric precompact neighborhood of the identity. Choose a positive real $\eta < \varepsilon/2$ such that for any $x, y \in X$ if $\rho(x,y) \leq \eta$ then $\rho(\alpha_b(x), \alpha_b(y)) \leq$

$\varepsilon/2$ for every $b \in B$. Set $\varepsilon' = \eta^2\lambda(B)$. Note ε' does not depend on $U, \mathcal{F}$ or δ.

Assume that $\widetilde{\phi}, \widetilde{\psi} \in \mathrm{Map}(\rho, U, \mathscr{F}, \delta, L)$ are ε' close i.e.

$$\rho_2(\widetilde{\phi}, \widetilde{\psi})^2 = \int_F \int_M \rho^2(\alpha_f(\phi(p)), \alpha_f(\psi(p)))\, d\overline{\mathrm{vol}}(p) d\lambda(F) \leq (\varepsilon')^2 = \eta^4\lambda(B)^2,$$

then there must exist some element $b \in B$ such that

$$\int_M \rho^2(\alpha_b(\phi(p)), \alpha_b(\psi(p)))\, d\overline{\mathrm{vol}}(p) \leq \eta^4.$$

An application of Markov's inquality gives that

$$\rho(\alpha_b(\phi(p)), \alpha_b(\psi(p))) \leq \eta$$

for other than $\eta^2 \operatorname{vol}(M)$ points in M. We thus have

$$\rho_2(\phi, \psi)^2 = \int_M \rho^2(\phi(p), \psi(p))\, d\overline{\mathrm{vol}}(p) \leq (\varepsilon/2)^2 + \eta^2 \leq \varepsilon^2,$$

which says that ϕ and ψ are ε-close. Therefore if the maps ϕ and ψ are ε- separated, then the corresponding maps $\widetilde{\phi}$ and $\widetilde{\psi}$ must be ε'-separated. And, hence $N_\varepsilon(\mathrm{Map}(\rho, E', \mathscr{F}', \delta', \sigma)) \leq N_{\varepsilon'}(\mathrm{Map}(\rho, U, \mathscr{F}, \delta, L))$. This proves that $h^{meas}_\Sigma(\mu, \alpha_\Gamma) \leq \lambda(G/\Gamma)\, h^{meas}_{\widetilde{\Sigma}}(\mu, \alpha)$. □

Theorem 6.2.2. *Let $\Gamma < G$ be a lattice in a locally compact group G and λ be a Haar measure on G. Let Σ be a sofic approximation to Γ and $\widetilde{\Sigma}$ be the induced sofic approximation to G. Suppose $\alpha : G \times (X, \rho) \to (X, \rho)$ is a continuous action on a compact metrizable space (X, ρ) and $\alpha_\Gamma : \Gamma \times (X, \rho) \to (X, \rho)$ its restriction to Γ. Then*

$$h^{top}_\Sigma(\alpha_\Gamma) = \lambda(G/\Gamma) h^{top}_{\widetilde{\Sigma}}(\alpha),$$

where $\lambda(G/\Gamma)$ *is the covolume of the lattice* Γ.

Proof. Normalizing the measure λ we can assume that $\lambda(G/\Gamma) = 1$. The proof of previous theorem shows that

$$h_{\Sigma}^{top}(\alpha_\Gamma) \leq h_{\widetilde{\Sigma}}^{top}(\alpha).$$

We now show the reverse inequality $h_{\Sigma}^{top}(\alpha_\Gamma) \geq h_{\widetilde{\Sigma}}^{top}(\alpha)$. Given a finite set $E \subset \Gamma$ and positive reals $\varepsilon, \delta > 0$, we want to find a precompact neighborhood $U \subset G$ of the identity, a positive real $\delta' > 0$ and a way of associating to the each map $\widetilde{\phi} \in \mathrm{Map}(\rho, U, \delta', L)$ a corresponding map $\phi \in \mathrm{Map}(\rho, E, \delta, \sigma)$.

Considering the proof of the previous theorem, one would like to define $\phi(p) = \widetilde{\phi}(e, p)$ – but the equivariance estimates of $\widetilde{\phi}$ are averaged over the fundamental domain and do not hold pointwise and hence might not hold for the identity e. In order to work around this issue, we try to find points in the fundamental domain $f_p \in F$ close to the identity for which the pointwise estimates are good enough and therefore define the map $\phi(p) = \alpha_{f_p^{-1}} \circ \widetilde{\phi}(f_p, p)$.

Let $F_0 \subset F$ be a compact set with $\lambda(F_0) > \varepsilon^2/8$ and $B \subset F$ be a symmetric neighborhood of the identity inside F. Choose a precompact neighborhood of the identity $U \subset G$ with

$$V = F_0 B \cup BEB \subset U.$$

Suppose $0 < \zeta < \min\{\varepsilon^2/8\lambda(V), \lambda(B)/2|E|\lambda(V)\}$ is small enough such that if for any $x, y \in X$ the distance $\rho(x, y) \leq \zeta^{1/2}$ then $\rho(gx, gy) < \delta/2$ for every $g \in V$. Set $\delta' = \zeta^4$.

If $\widetilde{\phi} \in \mathrm{Map}(\rho, U, \delta', L)$, then because of its δ'-equivariance property we have the bound

$$\frac{1}{\lambda(V)}\int_V\int_M\int_F \rho^2(\widetilde{\phi}\circ L_g(f,p),\alpha_g\circ\widetilde{\phi}(f,p))\,d\lambda(f)d\overline{\mathrm{vol}}(p)\,d\lambda(g)\le\zeta^8.$$

Applying Markov's inequality we get a subset $M'\subset M$ with measure $\overline{\mathrm{vol}}(M')>(1-\zeta^4)$ and one has

$$\frac{1}{\lambda(V)}\int_V\int_F \rho^2(\widetilde{\phi}\circ L_g(f,p),\alpha_g\circ\widetilde{\phi}(f,p))\,d\lambda(f)\,d\lambda(g)\le\zeta^4,$$

whenever $p\in M'$. Since $\zeta^2<\lambda(B)$, another application of Markov's inequality implies the existence of $f_p\in B$ with

$$\frac{1}{\lambda(V)}\int_V \rho^2(\widetilde{\phi}\circ L_g(f_p,p),\alpha_g\circ\widetilde{\phi}(f_p,p))\,d\lambda(f)\ \le\zeta^2$$

for every $p\in M'$. And, finally one concludes that for every point $p\in M'$ there is a subset $V_p\subset V$ with $\lambda(V_p)>(1-\zeta)\lambda(V)$ and for every $g\in V_p$ we have the bound

$$\rho^2(\widetilde{\phi}\circ L_g(f_p,p),\alpha_g\circ\widetilde{\phi}(f_p,p))\le\zeta. \tag{6.3}$$

Observe that by the choice of ζ if $p\in M'$ and a subset $B'\subset BEB$ has the measure $\lambda(B')=\lambda(B)$ then

$$\lambda(B'\cap V_p)\ge\lambda(B)(1-1/|E|), \tag{6.4}$$

and also

$$\lambda(F_0f_p^{-1}\cap V_p)>1-\varepsilon^2/8. \tag{6.5}$$

Given a point $p \in M'$, define

$$\phi(p) = \alpha_{f_p^{-1}} \circ \widetilde{\phi}(f_p, p) \tag{6.6}$$

and for the points $p \notin M'$, define $\phi(p) = \widetilde{\phi}(e, p)$. We claim that $\phi \in \mathrm{Map}(\rho, E, \delta, \sigma)$.

Part 1: δ - equivariance

Given $\gamma \in E$ and point $p \in M'$, applying (6.4) to the balls $Bf_{\sigma(\gamma,p)}^{-1}$ and $B\gamma f_p^{-1}$ implies the existence of an element $b \in B$ such that $bf_{\sigma(\gamma,p)}^{-1} \in V_{\sigma(\gamma,p)}$ and $b\gamma f_p^{-1} \in V_p$. Therefore using the estimate (6.3) we see that

$$\rho^2(\widetilde{\phi}(b, \sigma_\gamma \circ p), \alpha_{b\gamma f_p^{-1}} \circ \widetilde{\phi}(f_p, p)) = \rho^2(\widetilde{\phi} \circ L_{b\gamma f_p^{-1}}(f_p, p), \alpha_{b\gamma f_p^{-1}} \circ \widetilde{\phi}(f_p, p)) \leq \zeta,$$

and also

$$\rho^2(\widetilde{\phi}(b, \sigma_\gamma \circ p), \alpha_{bf_{\sigma(\gamma,p)}^{-1}} \circ \widetilde{\phi}(f_{\sigma(\gamma,p)}, \sigma_\gamma \circ p)) \leq \zeta.$$

From these two bounds and our choice of ζ we thus have

$$\rho^2(\alpha_{\gamma f_p^{-1}} \circ \widetilde{\phi}(f_p, p), \alpha_{f_{\sigma(\gamma,p)}^{-1}} \circ \widetilde{\phi}(f_{\sigma(\gamma,p)}, \sigma_\gamma \circ p)) \leq (\delta/2)^2.$$

Since the above is true for every $p \in M'$, hence we conclude that

$$\begin{aligned}
\int_M & \rho^2(\phi \circ \sigma_\gamma(p), \alpha_\gamma \circ \phi(p)) \, d\overline{\mathrm{vol}}(p) \\
&= \int_M \rho^2(\alpha_{\gamma f_p^{-1}} \circ \widetilde{\phi}(f_p, p), \alpha_{f_{\sigma(\gamma,p)}^{-1}} \circ \widetilde{\phi}(f_{\sigma(\gamma,p)}, \sigma_\gamma \circ p)) \, d\overline{\mathrm{vol}}(p) \\
&\leq (\delta/2)^2 + \zeta^4 \leq \delta^2.
\end{aligned}$$

Therefore, we have

$$\mathrm{Map}(\rho, U, \delta', L) \subset \mathrm{Map}(\rho, E, \delta, \sigma).$$

Part 2: ε - separation

Using continuity of the action α, choose $0 < \varepsilon' < \varepsilon^2/4$ with the property that if for any $x, y \in X$ the distance $\rho(x, y) \leq \varepsilon'$ then $\rho(fx, fy) < \varepsilon/4$ for every $f \in F_0$. We show that if maps $\widetilde{\phi}, \widetilde{\psi} \in \mathrm{Map}(\rho, U, \delta', L)$ are ε - separated, the corresponding ϕ, ψ are ε' - separated.

Notice that we suppressed the dependence of f_p on $\widetilde{\phi}$ in the notation. Keeping it in mind, observe that if M_0' is the intersection of two M's corresponding to $\widetilde{\phi}$ and $\widetilde{\psi}$, one has $\overline{\mathrm{vol}}(M_0') > 1 - 2\zeta^4$. Also, let ϕ and ψ be defined as

$$\phi(p) = \alpha_{f_p^{-1}} \circ \widetilde{\phi}(f_p, p) \quad \text{ and } \quad \psi(p) = \alpha_{g_p^{-1}} \circ \widetilde{\psi}(g_p, p)$$

whenever $p \in M_0'$.

In order to get a contradiction, assume ϕ and ψ are not ε' separated, it means

$$\int_M \rho^2(\phi(p), \psi(p))\, d\overline{\mathrm{vol}} \leq (\varepsilon')^2.$$

Thus there is a subset $M'' \subset M_0'$ having measure $\overline{\mathrm{vol}}(M'') > 1 - 2\zeta^4 - \varepsilon' > 1 - \varepsilon^2/2$ and $\rho^2(\phi(p), \psi(p)) \leq \varepsilon'$ whenever $p \in M''$.

For any $p \in M''$ the inequality (6.5) implies the existence of a set $F_p \subset F_0$ having the measure $\lambda(F_p) > 1 - 2\varepsilon^2/8$ and F_p is a subset of two V_p's corresponding to $\widetilde{\phi}$ and $\widetilde{\psi}$. Also, applying (6.3) leads to the bounds

$$\rho^2(\widetilde{\phi}(f, p), \alpha_f \circ \phi(p)) = \rho^2(\widetilde{\phi} \circ L_{ff_p^{-1}}(f_p, p), \alpha_{ff_p^{-1}} \circ \widetilde{\phi}(f_p, p)) \leq \zeta$$

and

$$\rho^2(\widetilde{\psi}(f,p), \alpha_f \circ \psi(p)) \le \zeta$$

whenever $f \in F_p$. Therefore, we can estimate the distance between $\widetilde{\phi}$ and $\widetilde{\psi}$,

$$\begin{aligned}
&\int_M \int_F \rho^2(\widetilde{\phi}(f,p), \widetilde{\psi}(f,p)) d\lambda(f) d\overline{\text{vol}}(p) \\
&\qquad \le \int_{M''} \int_{F_p} (2\zeta^{1/2} + \rho(\alpha_f \circ \phi(p), \alpha_f \circ \psi(p))^2 d\lambda(f) d\overline{\text{vol}}(p) + (3\varepsilon^2/4) \\
&\qquad \le (2\zeta^{1/2} + \varepsilon/4)^2 + 3\varepsilon^2/4 \le \varepsilon^2,
\end{aligned}$$

and conclude that if the maps $\widetilde{\phi}$ and $\widetilde{\psi}$ are ε- separated then, the corresponding maps ϕ and ψ must be ε'-separated.

From the two parts we thus have $N_\varepsilon(\text{Map}(\rho, U, \delta', L)) \le N_{\varepsilon'}(\text{Map}(\rho, E, \delta, \sigma))$ which proves $h(\alpha, \widetilde{\Sigma}) \le h(\alpha_\Gamma, \Sigma)$. □

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